

# OPERATIONAL AMPLIFIERS THROUGH EXPERIMENTATION AND REFLECTION

An Engineer's  
Journey

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Sencer Yeralan

OPERATIONAL AMPLIFIERS  
THROUGH  
EXPERIMENTATION AND REFLECTION  
*An Engineer's Journey*

Sarajevo, 2023





Vacuum tube operational amplifier, GAP/R K2-W, 1952

*Operational Amplifiers  
through  
Experimentation and Reflection  
An Engineer's Journey*

Sencer Yeralan, P.E., Ph.D.

Dedicated to the loving memory of Erdiñ Sayan.

1955 – 2019

## FOREWORD

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As a scholar of literature and the humanities, I surely am an outlier even among the “heterogeneous group of readers” that Professor Yerlan imagines for this text. It has been enlightening for me, however, to see confirmed here what I have long suspected from the stimulating and wide-ranging conversations that I have enjoyed with the book’s author over a number of years: we are both engaged, at the level of our subject matter, with the question of systems, their functioning and, as my particular academic inclination might insist, their necessary dysfunction. While the author’s childhood preoccupation with Tinker Toys led him to opamps and beyond, it now occurs to me, thanks to the clever framing of this text, that it is perhaps my own early exposure to Meccano that contributed to my understanding of texts as dynamic, diachronic systems. Primarily, it has been a pleasure, in reading this book, to recognize with fondness and respect the wise and humane voice of my good friend, esteemed colleague and fellow concerned pedagogue as he wittily engages and cheerfully accompanies his students and readers on their journey of initiation into his particular clan within the larger academic community.

Trevor John Hope  
Izmir, March 2023

## PREFACE

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In the late 80s, while I was working as a design engineer, I heard a cabinet maker say “you must standardize to customize”. I am sure he was not the one who coined the phrase, but nonetheless, I heard it first from him. I played with Tinker Toys and Erector sets when I was a kid. I also knew that Lego was big – not quite exciting as my Erector set, but fairly rich in its offerings. Erector sets always gave me a sense of engineering – literally nuts and bolts engineering. Whereas Lego seemed like the Lincoln Blocks some friends had.

To customize, you standardize – and operational amplifiers, affectionately called *opamps*, are perhaps the best example of this to an electronics engineer. You can make all sorts of things with opamps. Back then, it was the building block of analog computers – you know, the ones that had fancy cable connections reminiscent of the old telephone plug-boards. With these, you could integrate, differentiate, add, and subtract, and solve all sorts of differential equations. We liked differential equations – we thought they were the language of nature.

Later, as I started designing electronics systems for industry, I was often the “go-to” person to come up with analog circuitry. It is amazing what miracles a few transistors could do. Plus, using transistors is a bit like solving puzzles. It is both a challenge and an urge to seek elegance. If you design a circuit that does the job, and is nothing like you have seen before, there is an undefinable, inexplicable buzz, the satisfaction of which would not be traded for anything. I see the students missing out on that – I imagine there is no buzz if you find somethings on Wikipedia, sort of similar, not quite, but enough to forcefully shoehorn in as a surrogate solution.

But there was another way. Starting in the 80s, opamps became quite robust and capable, as their process went lower than a snake’s belly. These had such high performance for pennies, to us seasoned engineers, they seemed pure magic. Rail-to-rail operation, low noise, no offset or thermal drift to speak of, high frequency-gain suitable for almost RF – wow. Now why would anyone use discrete transistors when you could use an opamp? I held on perhaps for reasons of nostalgia, or a puritanical fixation to be content with modesty on the board. Perhaps I wanted to show off, “see how I can still design transistor circuits”.

But as prices dropped further, it started looking silly. When the cost of board space and assembly of discrete components became higher than opamps, the camel met its last straw.

This book was written as I taught an engineering course in circuits at the International University of Sarajevo. I tried to focus not only on opamps but add into the mix several relevant concepts. Since many non-electrical engineers took the course, it was important to frame the subject as a general engineering challenge, and to appeal to intuition. The hope is for the insights to carry over to other subjects of study as this heterogeneous group of readers would part their ways soon. I also wanted my students to see the human side of a practicing engineer. All with its shortcuts, rules of thumb, “*back-of-the-envelope*”s, and “*good enough*”s. The book follows step by step my class preparations. So it also embeds, to some extent, the contributions of the sophomore class of 2021. Parts where I had to revisit in subsequent class discussions naturally ended up covering more pages. The parts of the book skimmed over are the ones that class more readily understood and felt no further need for amplification. Most of it was written in real time, you might say, from late March to May, 2021.

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Sarajevo, June 2021  
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## ABSTRACT

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Engineering requires knowledge, skills, intuition, and pragmatism. It is more than a philosophical treatment of understanding. It is a craft. It takes time to master. And before you master, you must go through the early steps of apprenticeship. This all implies that you go over and over a topic, as each time you gain more understanding and your intuition deepens, and finally you feel comfortable with even the meta-engineering issues such as the societal or ethical.

A choice had to be made early on – should this be a complete treatise on operational amplifiers, or just a superficial first look, or somewhere in between? A complete in-depth treatise would require some background in transfer functions, systems modeling, complex calculus, etc. An intermediate book is perhaps the most difficult of all since it is context dependent. That is, a good understanding of what the reader knows and what is lacking becomes crucial.

This textbook pursues the other extreme. It is written as a practical guide to introduce the reader to operational amplifiers as an initial rather informal exposure. There is very little use of mathematics, and as such, it is suitable for anyone who has an interest in the topic. Our emphasis is on practicality and intuition rather than rigor. Those who are embarking on a career in electronics will undoubtedly go over the topics in this book in much more detail. Repetition is essential, as each time you go a bit deeper into the subject matter. This book aims to be a good first exposure. In addition, it may be used as a laboratory reference book for high school or university courses, or by hobbyists who would like to tinker with electronics and not necessarily go further.

The book is written in a conversational tone, as an experienced design engineer telling stories of days gone by dealing with volts, feedback, rain-dancing to make the circuits work, and smoke that was left in the chips by the manufacturers – smoke that had to be let out for proper operation.

*Comments and suggestions are welcome. Mail to sy@yeralan.org*

<https://www.yeralan.org>



*Be conservative in what you do, be liberal in what you accept from others.*

Jonathan Bruce Postel, September 1981

*Baseball is 90% mental and the other half is physical.*

Yogi Berra

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I would like to acknowledge my dear students, without whom, this book would not have been possible – or necessary.

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# CONTENTS

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## I Introduction

1	The Operational Amplifier	2
1.1	A First Look . . . . .	2
1.2	Idealization . . . . .	2
1.3	Let Us Build a Circuit . . . . .	4
1.4	Simulation – the Next best Thing . . . . .	6
1.5	A Useless Toy? . . . . .	7
1.6	The Secret Ingredient is <i>Feedback</i> . . . . .	9
1.7	A Bit of Intuition . . . . .	11
1.8	A Bit of Computation . . . . .	11
1.9	Tempered Gain . . . . .	12
1.10	Is All Feedback Negative? . . . . .	13
1.11	Experimentation and Reflection . . . . .	15

## II Basic Operations

2	Addition, Subtraction, Integration, and Differentiation	18
2.1	Golden Rules of Opamps . . . . .	18
2.2	Amplification . . . . .	19
2.3	Addition and Subtraction . . . . .	22
2.4	Integration . . . . .	23
2.5	Differentiation . . . . .	25
2.6	Superposition . . . . .	26
2.7	Experimentation and Reflection . . . . .	26
3	Analog Computers	28
3.1	An Example . . . . .	29
3.2	Industrial Control . . . . .	32
3.3	Experimentation and Reflection . . . . .	33

## III Signal Processing

4	Frequency and Gain	36
4.1	The Key Insight . . . . .	36
4.2	Integrator as a Low-Pass Filter . . . . .	41
4.3	Decibel – A Bit of History . . . . .	42
4.4	Revisiting the Integrator as a Low-Pass Filter . . . . .	45
4.5	A Practical Low-Pass Filter . . . . .	46
4.6	Active versus Passive Filters . . . . .	47

4.7	Experimentation and Reflection . . . . .	48
5	Higher-Order Filters	49
5.1	Not Steep Enough? . . . . .	49
5.2	Cookbook Recipes . . . . .	49
5.3	Band-Pass and Band-Reject Filters . . . . .	52
5.4	Some Generalization and Formalism . . . . .	54
5.5	A Bit of Magic . . . . .	56
5.6	Experimentation and Reflection . . . . .	58
<b>IV Formalisms</b>		
6	Phasors	60
6.1	Hints for a Mathematical Formalism . . . . .	60
6.2	A Nice Way to Visualize Phasors . . . . .	62
6.3	Euler's Identity to the Rescue . . . . .	63
6.4	The Impedance of a Capacitor . . . . .	64
6.5	Differentiation and Integration in Phasor Algebra . . . . .	66
6.6	Let Us Check . . . . .	66
6.7	A Tank Circuit . . . . .	71
6.8	Let us Build an Oscillator . . . . .	74
6.9	Experimentation and Reflection . . . . .	79
7	Transfer Functions	81
7.1	Transfer Functions . . . . .	82
7.2	Time for some Examples . . . . .	83
7.3	A Practical View of Transfer Functions . . . . .	86
7.4	Transfer Functions are Linear Operators . . . . .	89
7.5	Synthesis – Circuit Design . . . . .	90
7.6	The Laplace Transform . . . . .	95
7.7	Experimentation and Reflection . . . . .	98
<b>V Control Engineering</b>		
8	Control Engineering	100
8.1	Device Models . . . . .	100
8.2	Simulating Device Models . . . . .	102
8.3	Process Control . . . . .	104
8.4	Removing The Steady-State Error . . . . .	110
8.5	A Second-Order Plant . . . . .	112
8.6	Tuning Controllers . . . . .	116
8.7	Experimentation and Reflection . . . . .	117
<b>VI Opamp Applications</b>		
9	Colorful Opamp Circuits	120
9.1	Some Workhorse Opamp Circuits . . . . .	121

9.2	Power Circuits . . . . .	128
9.3	Clever Circuits . . . . .	130
9.4	Experimentation and Reflection . . . . .	137
10	Epilogue	139
	Bibliography	141
	Index	142
	Colophon	143

## LIST OF FIGURES

---

Figure 1.1	Simple Opamp Circuit. . . . .	4
Figure 1.2	Battery-Powered Split Power Supply. . . . .	5
Figure 1.3	QUCS Circuit. . . . .	6
Figure 1.4	Simulation Output Plot. . . . .	7
Figure 1.5	Light Detector. . . . .	8
Figure 1.6	Temperature Detector. . . . .	8
Figure 1.7	Simple Opamp Circuit with Feedback. . . . .	9
Figure 1.8	Simulated Opamp Circuit with Feedback. . . . .	10
Figure 1.9	Simulation Output of the Opamp Circuit with Feedback. . . . .	11
Figure 1.10	Voltage Follower. . . . .	13
Figure 1.11	Pump Driver with Hysteresis. . . . .	15
Figure 2.1	Inverting Amplifier. . . . .	20
Figure 2.2	Non-inverting Amplifier. . . . .	21
Figure 2.3	Summer. . . . .	22
Figure 2.4	Integrator. . . . .	23
Figure 2.5	Integrator Simulation. . . . .	24
Figure 2.6	Integrator Simulation Output. . . . .	25
Figure 2.7	Differentiator. . . . .	26
Figure 3.1	The PACE-TR-10 Analog Computer. . . . .	28
Figure 3.2	An Analog Computer Circuit. . . . .	30
Figure 3.3	Analog Computer Output. . . . .	31
Figure 4.1	Integrator. . . . .	36
Figure 4.2	The Effect of Capacitance on Gain. . . . .	37
Figure 4.3	The Effect of Frequency on Gain. . . . .	38
Figure 4.4	Integrator Lag. . . . .	40
Figure 4.5	The Effect of Frequency on Gain. . . . .	41
Figure 4.6	Low-Pass Filter. . . . .	46
Figure 4.7	Low-Pass Filter Frequency Response. . . . .	47
Figure 5.1	Low-Pass Filter Frequency Response of Two Cascaded Filters. . . . .	50
Figure 5.2	Second-Order Low-Pass Filter. . . . .	50
Figure 5.3	Second-Order Low-Pass Filter Frequency Re- sponse. . . . .	51
Figure 5.4	Second-Order High-Pass Filter. . . . .	51

Figure 5.5	Second-Order High-Pass Filter Frequency Response. . . . .	52
Figure 5.6	Band-Pass Filter. . . . .	52
Figure 5.7	Band-Pass Filter Frequency Response. . . . .	53
Figure 5.8	Band-Reject Filter. . . . .	54
Figure 5.9	Band-Reject Filter Frequency Response. . . . .	55
Figure 5.10	Full Bridge Rectifier. . . . .	56
Figure 5.11	State Variable Filter Circuit. . . . .	57
Figure 5.12	State-Variable Filter Frequency Response. . . . .	57
Figure 6.1	The Voltage-Current Behavior of a Capacitor. . . . .	60
Figure 6.2	The Simplest RC Circuit. . . . .	67
Figure 6.3	The Next Simplest RC Circuit. . . . .	68
Figure 6.4	A <i>Tank</i> Circuit. . . . .	72
Figure 6.5	The <i>Tank</i> Circuit, $V_A$ as a Function of Frequency. . . . .	73
Figure 6.6	A Simple Oscillator. . . . .	74
Figure 6.7	The Output Voltage of the Simple Oscillator. . . . .	74
Figure 6.8	The Voltage Divider Subcircuit. . . . .	75
Figure 6.9	A Simple Oscillator. . . . .	78
Figure 6.10	The Output Voltage of the Simple Oscillator. . . . .	78
Figure 6.11	A Series Tank Circuit. . . . .	79
Figure 6.12	The Sallen-Key Topology. . . . .	80
Figure 7.1	The Circuit as a Black Box. . . . .	82
Figure 7.2	The Circuit as a Black Box in the Transform Domain. . . . .	83
Figure 7.3	Low-Pass Filter. . . . .	84
Figure 7.4	R-C Voltage Dividers. . . . .	87
Figure 7.5	L-C Voltage Dividers. . . . .	88
Figure 7.6	Tank Circuit Voltage Dividers. . . . .	89
Figure 7.7	Cascading Transfer Functions. . . . .	89
Figure 7.8	Adding Transfer Functions. . . . .	90
Figure 7.9	A Designed-to-Order Circuit. . . . .	94
Figure 7.10	Frequency Response of the Designed Circuit. . . . .	95
Figure 8.1	Charging a Capacitor. . . . .	103
Figure 8.2	Voltage on a Capacitor being Charged. . . . .	104
Figure 8.3	Control Circuit (Excitation is Proportional to Error). . . . .	105
Figure 8.4	Control Response (Excitation is Proportional to Error). . . . .	106
Figure 8.5	Closed-Loop Control. . . . .	107
Figure 8.6	Equivalent System. . . . .	108
Figure 8.7	The Controller and the Plant. . . . .	108



Figure 8.8	The Proportional-Integral Controller and the Plant. . . . .	110
Figure 8.9	Proportional and Integral Control Circuit. . . . .	111
Figure 8.10	Proportional and Integral Control . . . . .	112
Figure 8.11	A Second-Order Plant Model. . . . .	113
Figure 8.12	A Second-Order Plant Model with a Proportional Controller. . . . .	113
Figure 8.13	A Second-Order Plant Model with a Integral Controller. . . . .	114
Figure 9.1	A Current Sink. . . . .	121
Figure 9.2	Ideal Rectifier. . . . .	122
Figure 9.3	Maximum of the Inputs. . . . .	122
Figure 9.4	Peak Voltage Detector. . . . .	123
Figure 9.5	Simple Fuzz. . . . .	123
Figure 9.6	The Effect of the <i>Fuzz</i> Setting on Clippings. . . . .	124
Figure 9.7	An Astable Multivibrator. . . . .	125
Figure 9.8	The Output of the Astable Multivibrator. . . . .	126
Figure 9.9	Asymmetrical Astable Multivibrators. . . . .	127
Figure 9.10	The Output of Asymmetrical Astable Multivibrators. . . . .	127
Figure 9.11	Power Boost. . . . .	128
Figure 9.12	Power Boost with no Cross-Over Distortion. . . . .	129
Figure 9.13	A Practical Power Boost Stage. . . . .	129
Figure 9.14	A Multivibrator with a <i>Virtual Capacitor</i> . . . . .	130
Figure 9.15	The Output of the Multivibrator with a <i>Virtual Capacitor</i> . . . . .	131
Figure 9.16	A <i>Virtual Capacitor</i> and its Equivalent. . . . .	132
Figure 9.17	A <i>Zen View</i> of the <i>Virtual Capacitor</i> . . . . .	133
Figure 9.18	A <i>Virtual Inductor</i> and its Equivalent. . . . .	134
Figure 9.19	Generalized <i>Virtual Load</i> . . . . .	135
Figure 9.20	A Negative Impedance Converter and its Equivalent. . . . .	136
Figure 9.21	A Negative Impedance Converter Experiment. . . . .	138

LIST OF TABLES

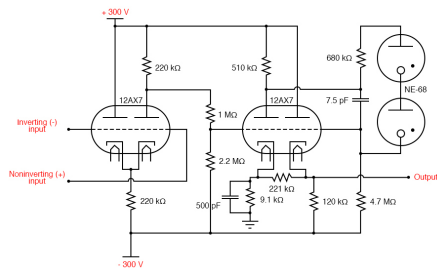
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Table 2.1	Characteristics of an Ideal Opamp. . . . .	19
Table 6.1	$V_A$ as a Function of Frequency $f$ . . . . .	68
Table 7.1	Laplace Transforms – Basic Survival Pack . . . . .	97

## Part I

### INTRODUCTION

What is an opamp? How to conceptualize opamp functionality. Simple circuits, baby steps.



GAP/R K2-W vacuum tube operational amplifier, 1952

## THE OPERATIONAL AMPLIFIER

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### 1.1 A FIRST LOOK

Electronics deals with information expressed and processed as some quality of electric flow, typically voltage. In common applications, it is quite straightforward to convert other attributes of electricity to voltage. The simplest example would be how current through a resistor creates a proportional voltage drop. Operational amplifiers, or as we will abbreviate throughout the book, *opamps* are best viewed as building blocks that deal with voltage. In ideal opamp has two inputs and one output. The inputs are referred to as the *non-inverting input* and the *inverting input*. These are often marked by (+) and (-) respectively. The output of an ideal opamp takes the difference of the voltages of its two inputs and amplifies this voltage by infinite gain. Let the voltage at the non-inverting input be  $V_+$  and at the inverting input be  $V_-$ . Let  $V_{\text{out}}$  denote the output voltage. An ideal opamp generates an output following the relation,

*The term “operational amplifier” is more conventionally abbreviated as “op amp”, that is, as two words. We will simply concatenate and say opamp*

$$V_{\text{out}} = G_{\infty}(V_+ - V_-) \quad (1.1)$$

where  $G_{\infty}$  denotes infinite gain.

### 1.2 IDEALIZATION

Clearly, an ideal opamp is indeed an *idealization*. Most obviously, there is no such thing as infinite gain. A typical opamp may have a gain at least six orders of magnitude. But  $10^6$  is not infinity. Another idealization, perhaps not immediately obvious, is that the output cannot exceed the supply voltages available in the circuit. There must be some voltage range available to power the circuit, say  $\pm 12\text{V}$ . Whatever equation 1.1 says, the output will most likely be somewhere in this range. That is, with a gain of a million, and  $V_+ = 0.5\text{V}$  and  $V_- = -0.5\text{V}$ , equation 1.1 predicts an output voltage of 1,000,000V. This clearly is not happening. What happens is that the output will be as high as

possible until it hits the supply voltage (or maybe somewhere a bit below that). We would say that the opamp is saturated. Let us dwell on another aspect of the idealization. It seems no matter how small the voltage difference between the inputs are, the opamp will sense the difference and amplify this value. There probably is a threshold of sensitivity, which we pretend is not there. For practical purposes, we may be forgiven, but nonetheless, this is an idealization. We also ignore any noise that may be present in the inputs. Similarly, we may ignore changes in characteristics, say in gain, with temperature. Actually, if you want to get fancy, you may entertain the thought of what happens when an opamp is accelerated to relativistic speeds. More realistically, down to Earth, though, there must also be (and of course, is) a time limit. That is how fast can an opamp react. Another way of asking this is “Is there a frequency limit to the input signals?”. And the answer is “Yes, of course.” Often, the frequency and gain are considered to be coupled, as we speak of the gain-times-frequency of an opamp. One may say, that  $10^6$  is the gain-times-frequency. So when the input frequency is, say 10, then the gain will be reduced by the same factor and be  $10^5$ . The good news is that for simple applications, say audio-frequency-range applications, or DC motor applications for mechatronics, a self-balancing-robot, for example, even a run-of-the-mill opamp is plenty fast and gainful enough.

Is that all? Well, not really. There are several other aspects which are not ideal. Let us name a few more.

We said the output voltage is a large multiple of the difference of the inputs. How linear is this? Well, for simple applications, we assume it is linear enough. Similarly, we assume that the output is capable of driving some load. How much power could the output supply? Typically, not very much. The output impedance of a typical opamp would be in the 100 Ohm range. Still, this is plenty for signal processing. There are opamps, called *power operational amplifiers* that are capable of supplying several Amps of current. Some are used as audio amplifiers or linear DC motor drivers. On the inputs side, we assume that the impedance at each input is infinitely high. That is, the inputs sense voltages without loading the signal sources. Typically, the input impedances are at least a megaOhm or more, higher when FET input stages are implemented. Most modern opamps are made with MOSFET circuitry, so the input impedances are indeed quite high.

## 1.3 LET US BUILD A CIRCUIT

Pick up your breadboard and start building a circuit. For this, we will use the good-old 741 opamp. There is good history of this industry workhorse. You may, of course use another opamp, but the basic principles will hold the same.

Here is the circuit diagram we will use. The pinout is given for the 8-pin DIP (Dual-Inline-Package) version of the chip. Note that there are two pins we do not use. These are to compensate for any null offset errors. When there is zero difference between the input voltages, the opamp output is not always zero, because of the minute differences among the input circuits due to manufacturing tolerances. So, this offset may be *nulled* by external compensation resistors. The offset is too small to make a difference in our example. For the time being, we just leave these pins unconnected. Also note that the package has pin 8, which is internally not connected to anything (designated in data books as an *NC pin*).

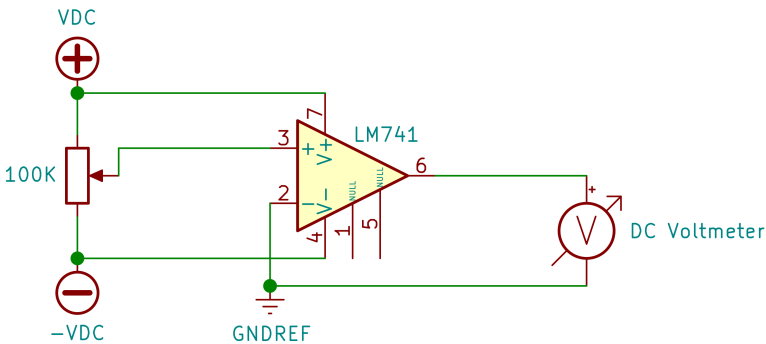


Figure 1.1: Simple Opamp Circuit.

You need a split-voltage lab power supply. Set the voltages to be around 9-12 V – DC of course. The mid voltage is called the reference or ground. If you like, you may use two batteries and make a simple power supply, sufficient to run the circuits below.



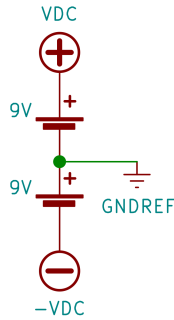


Figure 1.2: Battery-Powered Split Power Supply.

In the circuit, see how the opamp is supplied by the split voltages. The output is connected to a voltmeter. The output voltage is once again measured with respect to the reference or ground voltage. The inverting input too is connected to the reference or ground. Finally, a potentiometer is placed between the plus and minus outputs of the split voltage supply. Its wiper terminal is connected to the non-inverting input of the opamp. As you tweak the potentiometer, the voltage applied to the opamp will vary the full range of voltages, from the negative supply to the positive supply.

What do you expect to see?

Since the opamp output is a large gain times the difference between the voltages seen at the non-inverting and inverting inputs of the opamp, while the potentiometer wiper voltage is above the reference (ground) the opamp output should be high. Conversely, when the wiper voltage is below the reference voltage, the opamp output should be low. How high and how low? Well, since the gain is very high, and adjusting the potentiometer just right is practically impossible, you may expect the output to be saturated. That is as high as, or as low as, the opamp can possibly make it. Try to get an intermediate voltage. If you can, it would mean that you have adjusted the potentiometer just right so that the wiper voltage is (almost) exactly equal to the reference voltage. Expect the potentiometer to be very sensitive, why? Well, because the opamp has very high gain. The smallest perturbation of the potentiometer will disturb the very fine adjustment, and the output will be saturated high or low.

## 1.4 SIMULATION – THE NEXT BEST THING

If you cannot build the circuit, there is a way to gain considerable insights by using a circuit simulator. We use QUCS which is an open-source simulator that has all the features we need<sup>1</sup>. It works quite well on Linux. Nowadays there are many choices when it comes to circuit simulators. They are all quite user-friendly. You draw the circuit and simply run a simulation. QUCS uses the SPICE package of your choice. But generally, the default configuration is all we need. Figure 1.3 shows the simple circuit drawn in QUCS.

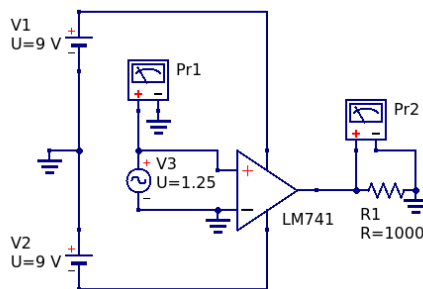


Figure 1.3: QUCS Circuit.

The 741 opamp is powered by two 9V supplies. A 1k resistor is placed as the output load of the opamp. Instead of a potentiometer, this circuit uses an AC voltage source to simulate the back and forth tweaking of the potentiometer. The AC voltage source is set to have a frequency of 10 Hz and an amplitude of 1.25V. There are two voltmeters, called *probes* in QUCS. Probe Pr1 measures the voltage applied to the opamp, while Pr2 measures the output voltage of the opamp.

Running the simulator, we may graph what the probes “see”.

The blue line in the graph (Figure 1.4) corresponds to Pr1, showing the input signal. As seen, the frequency is indeed 10 Hz, since each cycle takes 0.1 seconds. The red line shows what Pr2 sees at the output. As seen, the high gain of the opamp means that the output is quite quickly saturated, hitting the upper and lower bounds of what the opamp can muster.

<sup>1</sup> QUCS home page: <http://qucs.sourceforge.net/>

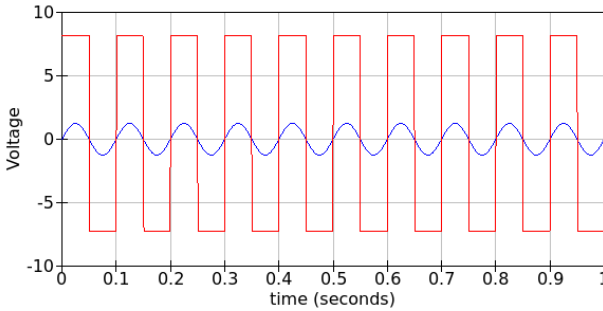


Figure 1.4: Simulation Output Plot.  
(red: opamp output voltage, blue: input voltage)

## 1.5 A USELESS TOY?

Well, trying to balance the potentiometer of [Figure 1.1](#) might be fun, but the circuit is really not that useful, other than verifying that the output is indeed a large gain times the differences in input voltages. Or is it?

Well, actually, the circuit is at the heart of many opamp circuits. In the parlance of electronics, it is called a *voltage comparator*. Say you want to detect if the fuel level is low and turn on a warning light. Or, if there is enough light, so that you want to turn off your outdoor patio lights. Or some device has overheated and you want to trip a fuse. You get the picture.

Take a look at the circuit [1.5](#). LDR is a *light-dependent resistor* whose resistance is high (say 100k) in the dark, but low (say 1k) when there is light. Along with the 10k resistor, it forms a voltage divider to supply the inverting input of the opamp. As light shines, the voltage at the inverting input drops, so the opamp output is high. The non-inverting input is connected to our trusted old potentiometer. Here, you may set the reference voltage, or the threshold. This way, you can trigger the output at a certain light intensity.

You must have noticed that rather than specifying the exact component we now use a generic diagram to represent the opamp. Nor did we show the pin numbers or the power pins. This is done to reduce clutter. When you perform your experiments, you will need to take care to make the right connections and provide the split-voltage power supply. We will do this from now on to reduce clutter in our circuit diagrams. In practice, the exact version of the opamp will depend on

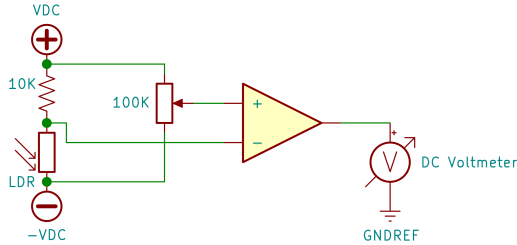


Figure 1.5: Light Detector.

many considerations, such as supply voltage, gain, input and output impedances, packaging, etc. As seen, the circuit is essentially a voltage comparator. Many an opamp end up with a lifelong career as voltage comparators. Here is another example: circuit 1.6 uses a thermistor. The resistance of a thermistor changes with temperature. So, in a sense, it is just like an LDR. To be more specific, this thermistor is a *negative thermal coefficient* one, that is, its resistance drops as it gets hot. There are also positive temperature coefficient thermistors. Curiously, there are no zero temperature coefficient thermistors. I wonder why...

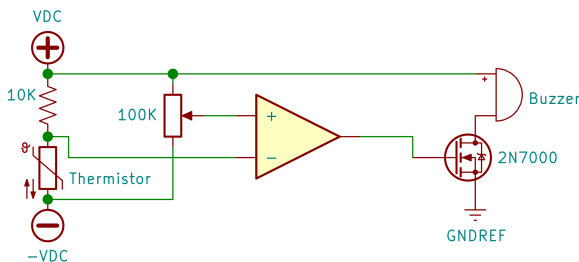


Figure 1.6: Temperature Detector.

Does the fact that the gate of the transistor could be close to the negative supply voltage (say  $-15\text{V}$ ) while the source is at GND? Well, you may inset a clamping circuit to make sure the gate does not fall below the source. This means a few extra components. A glance at the 2N7000 datasheet shows that the transistor safely operates with a gate-source voltage of  $\pm 20\text{V}$ . So, no need to panic or feel sorry for the transistor – it is tough enough to handle it.

Circuit 1.6 gets a bit fancy as the output drives a buzzer through a small N-channel enhancement mode MOSFET. I have used the 2N7000

*If this did catch your eye and somewhat bothered you emotionally, congratulations, you have the compassion to become a good electronics engineer.*

in many a circuit as drivers. The circuit starts looking busier, but with a trained eye, you should have no trouble seeing the voltage comparator that forms the basis of the device. Having such *circuitry X-ray vision* comes not only with experience, but also by paying attention to patterns and structures. If you start seeing structures and patterns, you are on your way to becoming an accomplished electronics engineer.

So, when will the circuit buzz? When the thermistor resistor drops, or when the temperature exceeds a threshold set by the potentiometer. Put this circuit in your refrigerator, and it will complain when you leave the door open longer than reason.

## 1.6 THE SECRET INGREDIENT IS FEEDBACK

Our very first circuit 1.1 is an example of an open-loop amplifier. It does not take much to saturate an opamp. The inputs are “*unaware*” of what the output is doing. Using information from the output to alter the input is the fundamental concept of *feedback*. The topic of feedback rather invites deep philosophical questions. After all, we often perceive feedback to have something to do with intelligence. A thermostat *knows* when to turn on the heat, and when to turn it off when the room is warm enough. The action (turning on the heater) is monitored by the thermostat, which turns the heater off at a certain temperature. This is the insight that leads us to the following modification.

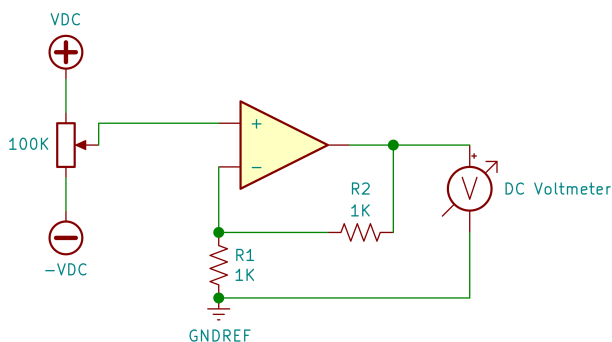


Figure 1.7: Simple Opamp Circuit with Feedback.

Now let us try to predict how the circuit will react. Once again, we have a voltmeter measuring the output voltage. The non-inverting input is connected to the potentiometer, as before, allowing us to sweep input voltages throughout the range of the split power supply. However,

the inverting input is not simply connected to the reference ground. It is connected to R1 and R2. Think of R1 and R2 as a voltage divider. Since their values are the same, the inverting input will see a voltage exactly half of the output voltage measured by our voltmeter. So far, so good.

*We will be talking about voltage dividers a lot in this book.*

Now let us say we set the potentiometer to put 1V to the non-inverting input. The opamp will react. If the inverting input voltage is lower than 1V the opamp output will rise. But how far? Will it saturate as before, to 12V or so? No. If it did the inverting input would see a voltage of 6V. Since then the voltage at the inverting input of the opamp is higher than that at the non-inverting input, the opamp will react by reducing the output voltage.

A little reflection will convince you that, when the voltage at the non-inverting input is 1V, so will the voltage at the inverting input. This means that the output will be at 2V. In fact, the output of the opamp will settle at twice the voltage at the non-inverting input. Any higher, then the opamp will reduce the voltage, and any lower, the opamp will increase the voltage. So the feedback loop is a corrective one. Corrective feedback is also known as *negative feedback*.

It is best if you build the circuit and examine how it works. If that is not an option, the next best thing is to simulate the circuit. Here we have the circuit as it was implemented in QUCS. It is a simple extension of our previous circuit, [Figure 1.3](#). The input is an AC voltage source with frequency 10 and an amplitude of 0.5 V.

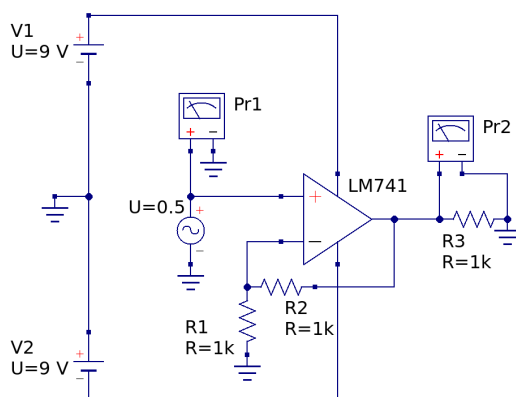


Figure 1.8: Simulated Opamp Circuit with Feedback.



When simulated, we may once again observe the input (the blue graph) and the output (the red graph).

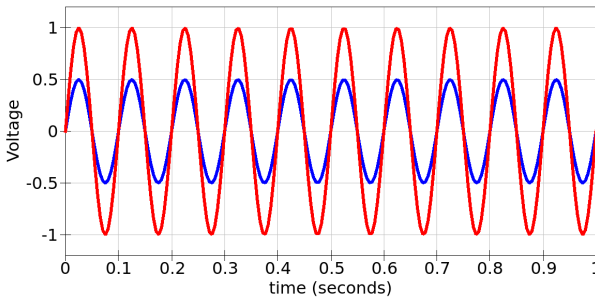


Figure 1.9: Simulation Output of the Opamp Circuit with Feedback.  
(red: opamp output voltage, blue: input voltage)

The simulator output is as expected. The output is synchronized with the input, but its amplitude is double that of the input. The output is exactly twice the input, is it not? You may also verify that the input is exactly 10 Hz.

## 1.7 A BIT OF INTUITION

When an opamp is run at a self-correcting stable manner, the voltages at the two inputs are almost identical. That is, the difference multiplied by a very high gain is a small voltage, between the  $\pm$  supply voltage, that we see at the output. For this to happen, the voltage difference must be that small value divided by the very large gain. Thus, we may think of the two inputs as having essentially the same voltage.

Negative feedback makes the circuit stable. With negative feedback, when the output overshoots, the response is to lower the output. Similarly, when the output is below the stable equilibrium point, the response is to increase the output. The response is always opposite to the error. Opposite, hence negative feedback.

## 1.8 A BIT OF COMPUTATION

Let us start with the circuit 1.7. Let  $R_1$  and  $R_2$  be arbitrary values. The voltage  $V_-$  at the inverting input will be

$$V_- = \frac{R1}{R1 + R2} V_{out} \quad (1.2)$$

Since we are convinced that  $V_-$  and  $V_+$  must be essentially equal,

$$V_+ = \frac{R1}{R1 + R2} V_{out} \quad (1.3)$$

or,

$$V_{out} = \frac{R1 + R2}{R1} = \left( 1 + \frac{R2}{R1} \right) V_{in} \quad (1.4)$$

Equation 1.4 perfectly describes the behavior of our circuit. But allow me to add one final comment. We often prefer to describe and think of the behavior of a circuit in terms of its so-called *transfer function*. We will use transfer functions extensively, especially as we combine subcircuits. Then the transfer function of the circuit will be evaluated as a function of the individual transfer functions of the subcircuits. The transfer function is simply the time-dependent output divided by the input. Why? Well then you may express the output as the input times the transfer function. In other words, the transfer function *transforms* the input to the output. Here the transfer function  $H(t)$  is not a function of time, but a mere constant.

$$H(t) = \frac{V_{out}(t)}{V_{in}(t)} = 1 + \frac{R2}{R1} \quad (1.5)$$

## 1.9 TEMPERED GAIN

It seems that the gain of the circuit is tempered, brought down from almost infinity to a constant of our choosing. Is it possible to make the gain of the circuit exactly unity (1)? Well, certainly. Notice that [Figure 1.10](#) may be viewed as [Figure 1.7](#) where  $R2 = 0$  and  $R1 = \infty$ . Then, according to [Equation 1.4](#), the output will be the same as the input. This circuit, where the gain is unity, is known as a *voltage follower*.

What, if any, you may wonder, is the use of a voltage follower – a circuit that seems to be no different from a simple wire that connects the output to the input. This is a good question, not because it is a difficult

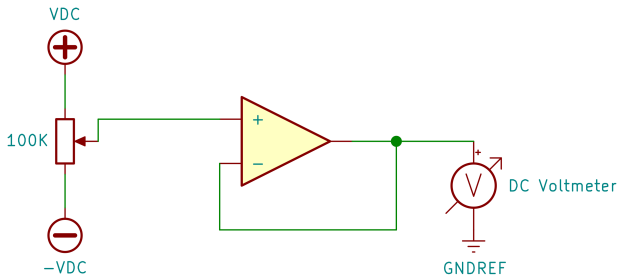


Figure 1.10: Voltage Follower.

one, but because it marks a certain milestone. If you can instinctively feel the answer to this question, you are on your way to becoming a good electronics engineer.

The typical simple answer has to do with impedance matching. Suppose you have a very sensitive sensor that puts out a voltage but very little current. If you would like to use the output of the sensor to drive a circuit, then you do not want to load the sensor. That is, you do not want to draw much current from the sensor. Another way of saying this is that the sensor has high output impedance. If you connect the sensor to a LED, for example, it will not have enough power to light the LED. It would practically be useless. If you use a voltage follower as in [Figure 1.10](#), then the opamp, with its high input impedance could read the voltage and provide an output suitable to light the LED. So, essentially, the voltage follower converts a high-impedance input to a low impedance output. You may think of connecting a microphone directly to a speaker, and see that it is futile to expect to hear anything from the speaker. The microphone (say a crystal microphone) has a high impedance, say 10 k $\Omega$ , whereas the speaker has a low impedance, say 8  $\Omega$ . The term *amplifier* in opamp fits, does it not?

## 1.10 IS ALL FEEDBACK NEGATIVE?

An opamp is indeed a versatile device. We saw that negative feedback has something to do with stability. The output swings opposite the the error, and thus always tries to reduce the error, whatever that may be. Here is a case that I faced. Not unlike the LDR and thermistor circuits, I had a circuit that ran a well pump when the pressure dropped

below a certain threshold. So, replace the thermistor with a pressure gauge, beef-up the output transistor, which in this case was driving a relay, and voilà, you have smart pump. Whenever the pressure drops the pump kicks in. Even if you do not use the water, pressure drops due to leaks, evaporation, etc. Now, here is the problem: the pressure drops below the threshold, fine. The pump kicks in, fine. After just a fraction of a second, the pressure builds up, and the pump kicks off once again, fine. But a little leak meant that the pressure immediately drops, ever so slightly, once again triggering the whole cycle over. The pump sounded like a misfiring engine, kicking in and out, causing wear and tear.

What we need is, in the parlance of electronics, hysteresis. This is a fancy word, borrowed from Greek, meaning something close to *lagging behind*. We want the pump to kick in at a certain pressure, say  $P_0$  and remain on until it reaches pressure  $P_1 > P_0$ , not immediately. Lag behind a bit, if you will. As the leak slowly reduces the pressure from  $P_1$  and  $P_0$ , the pump remains off. The same type of hysteresis is common in HVAC units, where there typically is a cut-off temperature and a trigger (cut-in) temperature, with a few degrees of difference between them.

Nor surprisingly, it is rather easy to incorporate hysteresis into voltage comparator circuits. Consider the one below (1.11). The pressure sensor output is fed into the inverting input of the opamp. If pressure drops, so does the sensor output voltage. When this voltage is below that seen at the non-inverting input, the opamp is triggered and the relay energized, thereby turning on the pump. When the pressure is high enough, the pressure sensor output exceeds the voltage seen at the non-inverting input, and the relay is switched off. Here is the critical point: what is the voltage seen at the non-inverting input of the opamp? Unlike previous circuits, we see that there are two resistors,  $R_1$  and  $R_2$  that form some sort of feedback, but this time, it involves the non-inverting input. Once again, we may consider  $R_1$  and  $R_2$  to form a voltage divider. There is a bit of engineering intuition needed here. Note that the reference potentiometer has a value of 1k, an order of magnitude lower than the feedback resistor  $R_1$  and two orders of magnitude lower than  $R_2$ . We may assume that the voltage at the wiper terminal of the potentiometer is not affected by the current flowing through the feedback resistors. In other words, we may assume that the reference set by the potentiometer is fixed throughout the operation of the circuit. Let this voltage be  $V_{ref}$ . Let the output of the opamp

be  $V_{hi}$  when saturated high, and  $V_{lo}$  when saturated low. We expect  $V_{hi}$  and  $V_{lo}$  to be approximately the split power supply voltages.

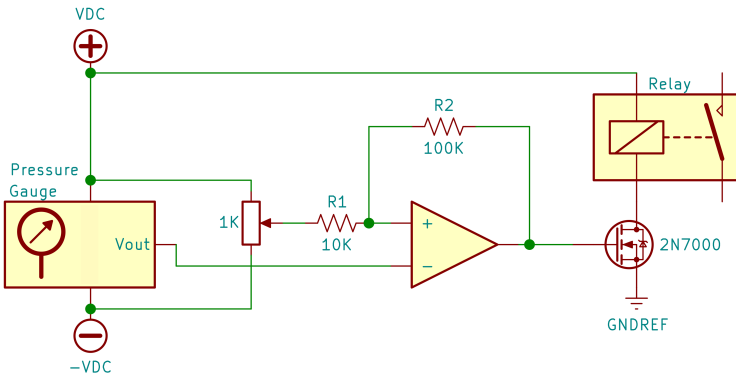


Figure 1.11: Pump Driver with Hysteresis.

When the opamp output is high, the non-inverting input is not at  $V_{ref}$  but  $V_{ref} + V_{hi} \frac{R1}{R1+R2}$ . This is a little over the reference threshold set by the potentiometer. For the pump to turn off, the pressure has to build up a little over that determined by  $V_{ref}$ . Once the pressure exceeds the threshold and the pump kicks off, the output is  $V_{lo}$ . Now the voltage seen at the non-inverting input is  $V_{ref} + V_{lo} \frac{R1}{R1+R2}$ . Since  $V_{lo}$  is negative, the trigger point for the pump to kick back in is lower than that set by  $V_{ref}$ . In short, the pump kicks off a little over the reference point and kicks back in a little under the reference point. The range is determined by  $\frac{R1}{R1+R2}$ . This is typically small enough not to create a noticeable pressure difference, but large enough so that the pump does not cycle on and off too often.

## 1.11 EXPERIMENTATION AND REFLECTION

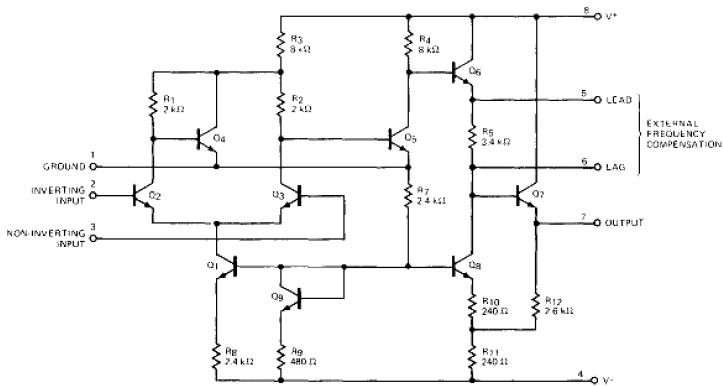
1. Referring to the circuit 1.7, try different values of  $R1$  and  $R2$  to see if relationship 1.4 holds.
2. Extend the concepts in circuits 1.5 to trigger the output when light is below a threshold, within a given range, or outside a given range. You probably need two potentiometers, one to set the upper limit of the range, and one to set the lower limit.
3. Repeat the same with thermistors, that is, extend the circuit 1.6. Try both positive and negative coefficient thermistors.

4. Write an equation in the spirit of 1.4 that explains the positive feedback hysteresis case. Also include cases where the potentiometer is not at least an order of magnitude smaller than the feedback resistors.
5. Would it ever make sense to have both positive and negative feedback in a voltage-comparator-type opamp circuit? If so, give an example.

## Part II

### BASIC OPERATIONS

Simple signal processing: addition, subtractions, integration, and differentiation. The bread and butter of all circuits.



Fairchild Linear Integrated Circuits  
 $\mu A 702$  Wideband DC Amplifier, 1963

## ADDITION, SUBTRACTION, INTEGRATION, AND DIFFERENTIATION

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Many opamps choose a career path that places them in circuits where they serve as general-purpose analog signal processors. I say *analog*, because there have been other ways for some time now. But, back in the day, we represented various time-varying physical quantities by a signal – mostly by the voltage of the signal. A simple example is sound. The sound waves and the signal that comes from the microphone are essentially the same in many ways. Say, a pure sinusoidal signal from the microphone would indicate a pure tone, which if we listened to, might sound like a nice flute. We may amplify, add, subtract, integrate, or differentiate these signals as we process them. A preamplifier that combines several audio inputs before sending the composite to a power amplifier essentially adds the signals, does it not? In the previous chapter we discussed voltage comparators. These take action based on the attributes of the signal. So, perhaps, it is not all that inappropriate even to call those analog signal processors.

In this part, we will delve into of the the basic signal processing tasks: amplify, add, subtract, integrate, and differentiate. But before we do so, let us first develop some methodology. Here, by methodology, we mean a general approach to not only how we think about the tasks, but some standardized formal solution procedures. These procedures will be based on a few starting points. This general structure is a formalism. We have basic premises (axioms of some sort) and rules of inference to reach desired conclusions.

### 2.1 GOLDEN RULES OF OPAMPS

When we think of ideal opamps, we assume a set of simplifying conditions.

Most useful of ideal opamp properties in understanding existing circuits and designing new ones are the summarized.

- A. Their input impedances are infinitely large.
- B. Their output impedances are relatively low.



Table 2.1: Characteristics of an Ideal Opamp.

Operation	
Open-loop gain	$\infty$
Operating frequency	DC to $\infty$ Hz
Chip-wide Characteristics	
Common mode rejection ratio	$\infty$
Power supply rejection ratio	$\infty$
Noise	non-existent
Temperature drift	none
Inputs	
Input impedance	$\infty$
Input voltage range	$-\infty$ to $\infty$
Input offset voltage	0 V
Outputs	
Output voltage range	$-\infty$ to $\infty$
Output impedance	0 ohm

- c. If the opamp is running in a stable manner (not saturated), then the voltage at the two inputs are essentially the same.

We are now ready to establish a formal approach to the analysis of opamp circuits, which will also help us in the design of such circuits. We will illustrate and gradually build up our comprehension by examples.

## 2.2 AMPLIFICATION

Consider the circuit ([Figure 2.1](#)) below. Once again, we have the resistors  $R_1$  and  $R_2$ , in a negative feedback configuration.

Let  $V_{in}$  and  $V_{out}$  be the input and output port voltages. These are shown as little tabs in the circuit. The non-inverting input is connected to ground. We are now ready to analyze the circuit. Our formalism goes like this:

- The opamp is run in a stable manner. That is, the output will be within the range of the supply voltages, not saturated high or low. A swing in the output voltage is fed back to the inverting

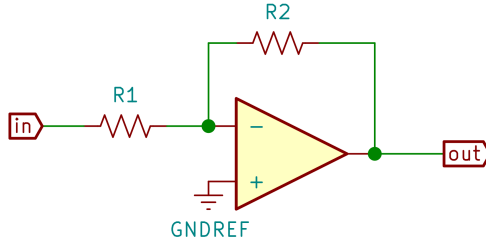


Figure 2.1: Inverting Amplifier.

input, so the response will be in the opposite direction of the error.

- By Golden Rule c, the inverting and non-inverting inputs will be at the same voltage level. This means that the inverting input is at ground reference (0V).
- Golden Rule A says that the input impedance is infinitely high, so no current actually enters or exits the opamp.

Putting all this together, the current from the input through R1 is the same as the current flowing through R2 to the output. The inverting input is at voltage zero. So,

$$\frac{V_{in} - 0}{R1} = \frac{0 - V_{out}}{R2} \quad (2.1)$$

or,

$$V_{out} = -\frac{R2}{R1} V_{in} \quad (2.2)$$

It seems to me that this is the cleanest opamp configuration. Easy to understand, with a simple equation that describes its behavior. A unity inverting amplifier is possible by choosing two resistors with the same value. Moreover, a non-inverting amplifier may be constructed by cascading two of these inverting ones. But, you probably would not do this, and rather just use a voltage follower.

Now, let us consider the non-inverting amplifier (Figure 2.2).

Following the same logical steps, realizing that the circuit is stable, the inverting input must be at the same voltage level as  $V_{in}$ . The voltage at the inverting input is the same as that determined by the voltage divider comprising R1 and R2. We have,

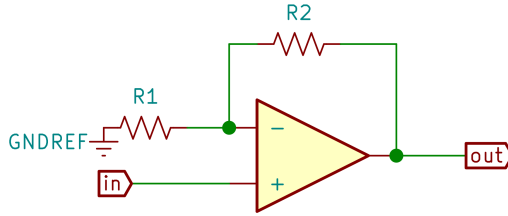


Figure 2.2: Non-inverting Amplifier.

$$\frac{R1}{R1 + R2}(V_{out} - 0) = V_{in} \quad (2.3)$$

or,

$$V_{out} = \frac{R1 + R2}{R1} V_{in} \quad (2.4)$$

or simply,

$$V_{out} = \left(1 + \frac{R2}{R1}\right) V_{in} \quad (2.5)$$

There we have it. Almost as nice as [Equation 2.2](#), but with an extra 1. When the gain,  $\frac{R2}{R1}$  is high, we may ignore the 1. The good thing is that the configuration does not invert the polarity of the signal.

Why do you think there is an additional 1 in the gain when we have the non-inverting configuration? Of course, you can verify by mathematics, but is there any other way we may interpret this? Is there any *engineering intuition* that would be helpful? Actually, I would answer in the affirmative. You see, if the feedback resistor is just a wire, zero Ohms, then the opamps will be voltage followers. The non-inverting configuration follows the input voltage, hence a gain of 1. The inverting configuration follows the voltage at the non-inverting input, which is zero. It does not look like much, but it is those little things that violate complete symmetry, albeit in a minute way, that provides a glimpse of how reality is actually not as perfect as we may model in our approximations. Sure, you may ignore the 1 when gain is sufficiently high, but there it is, nonetheless. The spice of life, and how lucky us engineers are, we get to appreciate it.

*This reminds me of the saga of charge, parity, and time reversal symmetry. Most thought that the laws of physics were symmetric in charge, parity, and time. Not so, apparently. Many argue that this is why there is more matter than anti-matter in the universe.*

## 2.3 ADDITION AND SUBTRACTION

Suppose we have two input signals which we would like to combine. That is, we want to add these signals and create a composite signal. This could be an audio mixer, for example. We may choose an inverting or a non-inverting configuration. Let us do the former (Figure 2.3).

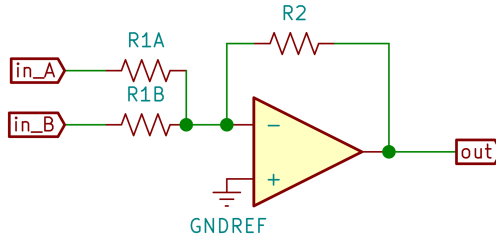


Figure 2.3: Summer.

Following the same logical steps given for Figure 2.1, we recognize that current flows into the inverting input from both resistors R1A and R1B, while the total current, once again, flows through R2. Modifying Equation 2.1, we have,

$$\frac{V_{in\_A} - 0}{R1A} + \frac{V_{in\_B} - 0}{R1B} = \frac{0 - V_{out}}{R2} \quad (2.6)$$

or,

$$V_{out} = - \left( \frac{R2}{R1A} V_{in\_A} + \frac{R2}{R1B} V_{in\_B} \right) \quad (2.7)$$

Subtraction is a bit like addition. We will not give an explicit description of a circuit that subtracts. It is left as an exercise (see Section 2.7).

Before we continue, let us pause and take in the meaning of the summer. I would submit that there is more here than meets the eye. You see, the total effect of two inputs is the same as the sum of their individual effects. If we built two separate circuits and input the voltages, then measured the outputs, we would find that the sum of the outputs of these individual circuits would be just the same as what we see with the summer amplifier. Sum of causes result in the sum of effects. This is known as “*superposition*”. It is a rather general property of linear systems, so much so that, we may be inclined to *define* a linear system as *one where superposition holds true*.

## 2.4 INTEGRATION

So far we used resistors in our circuits. When there is no hysteresis, the output voltage will always be determined by the input voltages. In other words, with only resistors, the current-voltage relationship is not time-dependent. We developed a nice formal approach to analyzing these circuits. The same principles and golden rules must hold when we consider time-dependent current-voltage relations. The circuit below is known as an integrator.

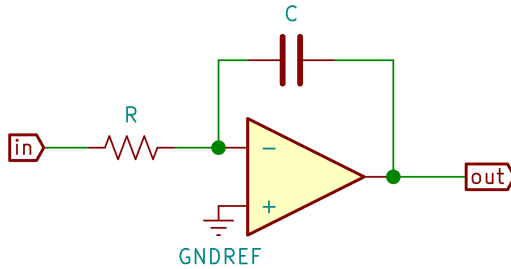


Figure 2.4: Integrator.

Once again, the same logical steps given for [Figure 2.1](#) apply. We recognize that current flows into the inverting input through resistor  $R$ , while this current flows out through  $C$ . The current through a capacitor is a function of the voltage across the capacitor. Let  $V_C(t)$  be the voltage across the capacitor, and similarly, let  $i_C(t)$  be the current that flows through the capacitor. We know that,

$$i(t) = C \frac{dV(t)}{dt} \quad (2.8)$$

Similar to the argument for [Equation 2.1](#), we have,

$$\begin{aligned} \frac{V_{in}(t) - 0}{R} &= C \frac{d(0 - V_{out}(t))}{dt} \\ \frac{V_{in}(t)}{RC} &= - \frac{dV_{out}(t)}{dt} \end{aligned} \quad (2.9)$$

$$V_{out}(t) = - \frac{1}{RC} \int_0^t V_{in}(\tau) d\tau$$

Let us make a few observations. First, if  $V_{in}(t)$  remains non-zero for a sufficiently long time, the integrator output will reach the supply voltages and the opamp will saturate. The integrator is thus meaningful when we have AC signals of suitably high frequency. Next, let us dwell a bit on the units. Studying the units is rather useful in developing engineering intuition.

The left-hand-side of the integral equation in [Equation 2.9](#) has units Voltage, denoted by [V]. We follow the convention from *dimensional analysis* and use square brackets to denote the units. The integral on the right-hand-side has units [V][T], where [T] is the unit of time. The constant RC must therefore have units [T]. This is indeed the case. The term RC is referred to as the *time constant*. It pops up quite frequently in electronics. You also see a trade-off between R and C. If you want a certain time-constant value, you may implement it by selecting various values of each, as long as their product is what you want. Typically, small capacitors and large resistors are used, since large capacitors may become bulky and age faster over the life of the product.

The best way to gain insights and sharpen your intuition would be to build the circuits and experiment with them. Once again, the worthy but not quite as powerful substitute to actually building the circuits would be to simulate. If you cannot build, at least simulate.

The circuit ([Figure 2.5](#)) was simulated in QUCS.

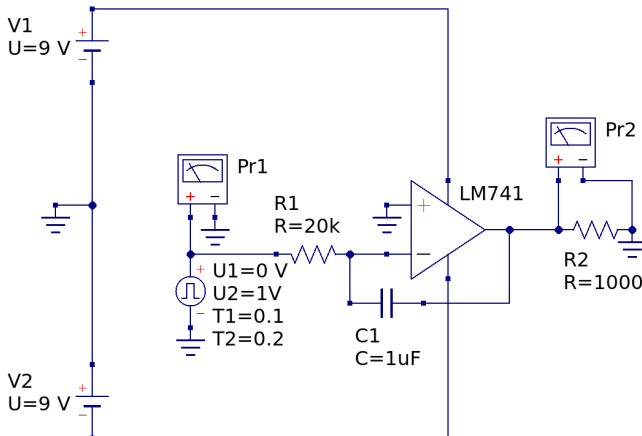


Figure 2.5: Integrator Simulation.

The integrator is built with a  $1\ \mu\text{F}$  capacitor and a  $20\text{k}$  resistor, which gives a time constant of  $0.02$  seconds. The excitation comes from a pulse generator with amplitude  $1\text{V}$ . It switches on at time  $0.1$  seconds, and switches off at  $0.2$  seconds. The pulse is  $0.1$  seconds long. The simulation runs give the expected output (Figure 2.6).

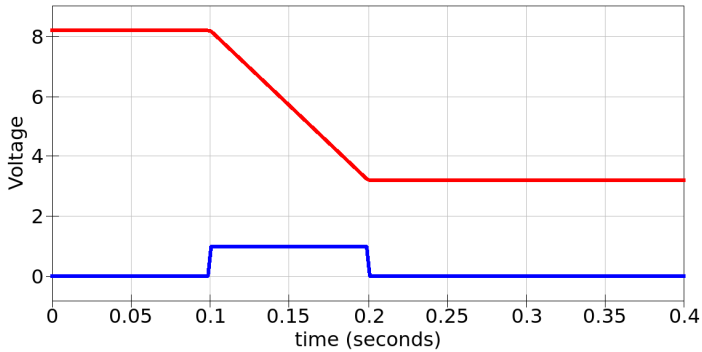


Figure 2.6: Integrator Simulation Output.

Note that the capacitor was charged at  $8\text{V}$  at the beginning of the simulation. The output remains at  $8\text{V}$  until the pulse is triggered at time  $0.1$  seconds. For the next  $0.1$  seconds, the pulse remains high. The output has a linear response, albeit with a negative slope. The negative slope is because the positive pulse is applied to the inverting input of the opamp. Now, the reciprocal of the time constant is  $50$ , and the integral of the signal is  $0.1$  ( $1\text{V}$  for  $0.1$  seconds). Thus the integral value is  $5\text{V}$ . This is what the graph shows. The voltage drops from  $8\text{V}$  to  $3\text{V}$ .

## 2.5 DIFFERENTIATION

Since integration seems quite possible, so must its opposite operation, differentiation. Indeed, switching the locations of the resistor and the capacitor would accomplish this (Figure 2.7).

The analysis follows the same logic.

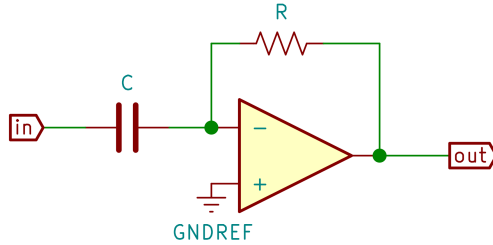


Figure 2.7: Differentiator.

$$C \frac{d(V_{in}(t) - 0)}{dt} = \frac{(0 - V_{out}(t))}{R}$$

$$\frac{dV_{in}(t)}{dt} = -\frac{V_{out}(t)}{RC} \quad (2.10)$$

$$V_{out}(t) = -RC \frac{dV_{in}(t)}{dt}$$

## 2.6 SUPERPOSITION

So far we used resistors in our circuits. When there is no hysteresis, the output voltage will always be determined by the

## 2.7 EXPERIMENTATION AND REFLECTION

1. Suppose we want to construct a non-inverting amplifier with gain 100 by cascading two inverting amplifiers as shown in [Figure 2.1](#). While the product of the gains must be 100, this product may be split up in many different ways. For example,  $-100$  and  $-1$ , or  $-10$  and  $-10$ , or  $-1$  and  $-100$ , etc. What would be your preference? Why? Discuss the pros and cons of each alternative.
2. How can the *summer* circuit [Figure 2.3](#) be generalized to accommodate an unspecified number of inputs?
3. How can the *summer* circuit [Figure 2.3](#) be generalized to subtract one signal from the other?



4. Is the integrator circuit stable? That is, will it settle to a known and desirable state when left alone with no input? How about the differentiator circuit?
5. Design and investigate the pros and cons of an integrator with a non-inverting configuration.
6. Design and investigate the pros and cons of a integrators using inductors, rather than capacitors.
7. Do the units in [Equation 2.10](#) match up?

## ANALOG COMPUTERS

Now that we have the means to amplify, add, subtract, integrate, and differentiate signals, what practical things can we do with opamps? Surprisingly, a lot. And there is quite a lot of history behind this. In fact, I was first introduced to opamps as an ambitious lad in my early 20s with analog computers. I vividly remember using the PACE-TR-10 and thinking it was the greatest thing ever. A few years later, when I started graduate school in 1978, the department had a course on analog computers, taught by Professor P. E. Valisalo. He was quite passionate about the topic and had many ideas [8]. As students, we were fairly saddened by the way he was treated by the newer, young and ambitious professors who insisted that the digital computer was the only way to go. Maybe that has something to do with why nowadays only a few people appreciate – or even understand – differential equations.

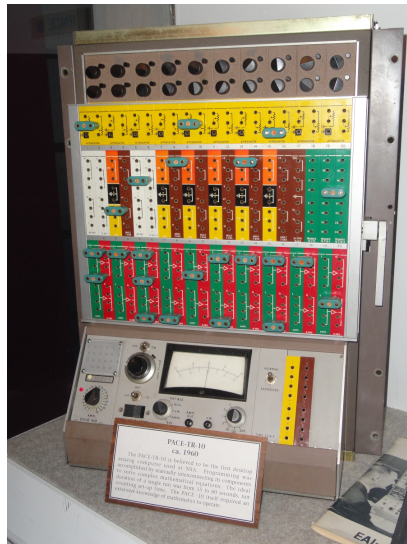


Figure 3.1: The PACE-TR-10 Analog Computer.

An analog computer is a device that uses signals, typically voltage, to represent time-varying physical quantities. You can implement and “solve” differential equations with analog computers. I say “solve” be-

cause you essentially observe the resultant waveform on an oscilloscope. Fancy ones let us plot these waveforms as well.

### 3.1 AN EXAMPLE

Perhaps it is best to illustrate the idea of analog computation with an example. Everyone knows how a little mass, say  $m$  suspended from a rubber band will bounce up and down when stretched and let go. Let us measure the distance of the mass from its resting point as variable  $x$ . Since the position, once disturbed, changes, we are interested in the time-dependent value of distance, which we will denote as  $x(t)$ . Now, comes the critical concept: let us construct a circuit, where the voltage at a certain node mimics the same time-varying response. In fact, I will also call this voltage  $x(t)$ .

Now recall some high school physics. The force on the rubber band due to the movement of the mass is  $m \frac{d^2x(t)}{dt^2}$ . The force exerted by the rubber band is proportional to its displacement from equilibrium. That is  $kx(t)$ , where  $k$  is the so-called *spring constant*. In addition, there are frictional forces, such as air resistance or internal friction within the rubber band. These forces are proportional to the velocity of the mass,  $f \frac{dx(t)}{dt}$ . And finally, we have the ever-so-present gravitational force  $mg$ . Putting it all together, we have the equation,

$$m \frac{d^2x(t)}{dt^2} + f \frac{dx(t)}{dt} + kx(t) = mg \quad (3.1)$$

This is a second-order non-homogeneous linear differential equation. Normalizing (dividing) by the mass, we have,

$$\frac{d^2x(t)}{dt^2} + \frac{f}{m} \frac{dx(t)}{dt} + \frac{k}{m} x(t) = g \quad (3.2)$$

Since this is a second-order differential equation, we will need two initial conditions to find a specific solution. Considering that at the beginning we will stretch the rubber band and let go, we may specify an initial position, say  $x_0$ , and assume that the initial velocity is zero.

$$\begin{aligned}
 x(0) &= x_0 \\
 \left. \frac{dx(t)}{dt} \right|_{t=0} &= 0
 \end{aligned}
 \tag{3.3}$$

As stated at the outset, we will represent  $x(t)$  with a voltage. Consider the circuit given in Figure 3.2.

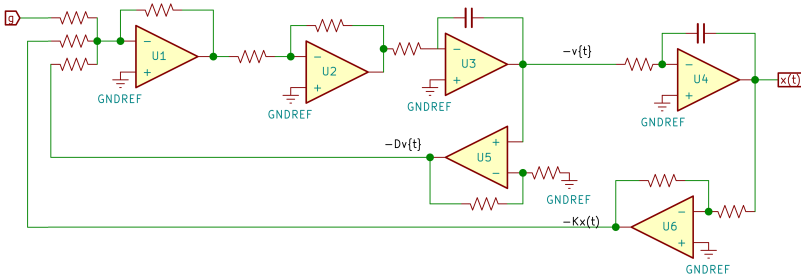


Figure 3.2: An Analog Computer Circuit.

The circuit seems a bit busy, consisting of six opamps, used as summers, amplifiers, and integrators. The component values are not explicitly given. The circuit is this more of a demonstration of concept. Let us follow the signals around the circuit and see how it implements the bouncing mass suspended from the rubber band. You will notice that there is a circular topology to the design, a bit like the ouroboros. So, we may start at any point around the circuit and trace the signals. Let us, though, start with opamp U1. We re-arrange the differential equation, so that the second derivative term is expressed as a linear combination of the variable, its first derivative, and a constant.



The Ouroboros

$$\frac{d^2x(t)}{dt^2} = -\frac{f}{m} \frac{dx(t)}{dt} - \frac{k}{m} x(t) + g
 \tag{3.4}$$

The implication is that we need not explicitly model the second derivative term, but be happy that we can combine other ingredients to express its value. The opamp U1 is used to add three different signals. These are the constant  $g$  which is input to the circuit, the negative of the first derivative term, and the negative of the term that includes

the variable  $x(t)$  itself. The two latter terms are to be multiplied by constants, the  $\frac{f}{m}$  and the  $\frac{k}{m}$ , before being added. These multiplications are done by U5 and U6, respectively. The output of U1 is inverted by U2. The output of U2 thus constructs the second derivative, the left hand side of (Equation 3.4), by combining the individual terms on the right hand side of that equation. This signal is fed to integrator U3 to produce the first derivative of the signal  $x(t)$ . Since the integrator also inverts the signal, it is actually  $-\frac{dx(t)}{dt}$ . You would see that this is one of the signal taps, feeding U5. U5 acts as an amplifier, whose output becomes  $-\frac{f}{m} \frac{dx(t)}{dt}$ , ready to be fed into the summer U1. We have completed one of the loops in the circuit. Opamp U4 integrates the signal, the first derivative of  $x(t)$ . The output of U4 is what we would observe, either on a scope or on a plotter. The voltage swings at the output of U4 will be the same as the swings in position of the original mass and rubber band system. The output of U4 is multiplied by  $-\frac{k}{m}$  by U6. This signal is the other input to the summer.

Once the circuit is set up, you may start the process by charging one of the capacitors, say the one in the feedback loop of U4. Then, observe the output of U4 to see how the system evolves. The various parameters, namely  $m$ ,  $f$ , and  $k$  are to be determined by the values of the resistors at each of the amplification opamps.

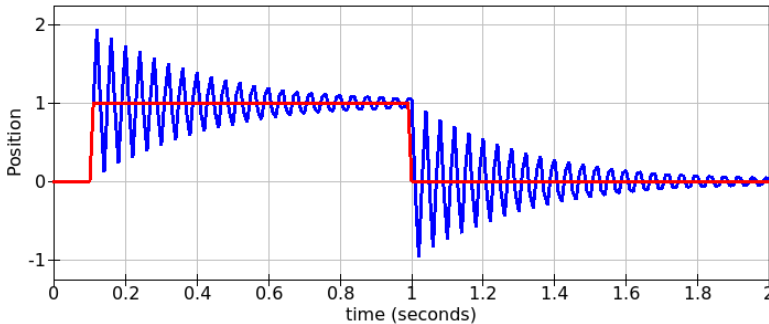


Figure 3.3: Analog Computer Output.

It is best to actually build the circuit and see how it behaves. If you simulate, you gain some insights, but lose the real engineering touch. The simulator output (Figure 3.3) shows the response. The horizontal axis is time in seconds. The vertical axis is position. Positive positions

are in the direction of  $g$ , that is, downward (yes, strangely, up is down here). The blue line is the output voltage  $x(t)$  corresponding to the position of the mass. The red line is the excitation. I chose to pulse the input  $g$ . That is, the mass is under gravitational force for a one second, and then the gravitational force is removed. I know, in reality, it is not easy to turn the gravitational force on and off. I suppose you could consider this to be a model of a piece of iron suspended by a rubber band over an electromagnet. Then you could turn the electromagnet on and off to make the mass bounce up and down. The analog computer correctly duplicates what happens to the mass. The position oscillates, but dampens over time to settle to an equilibrium state. When the gravitational force is removed, again after dampened oscillations, the position settles at zero.

There was a time when analog computers were used extensively to study the dynamics of systems, especially in the aviation industry. Nowadays, simulations are almost exclusively done on digital computers. But one cannot deny that building an analog computer circuit to mimic and simulate a dynamical system governed by differential equations has a place in engineering. If nothing else, it serves to elevate engineering intuition.

### 3.2 INDUSTRIAL CONTROL

It would be misleading to think that analog computers were the only place where opamps shone. Along with analog computers, much industrial control was accomplished by things that resemble analog computers, again constructed using opamp building blocks. A significant portion of such control falls under what is known as *PID control*. The term derives from the words *proportional*, *integral*, and *derivative*.

The idea is to express the process which is to be controlled by a signal, and continuously compare it against a reference signal. The reference signal may be time-dependent. A good example of time-varying reference voltage is a servomotor that needs to track some trajectory. The voltage fed to a servomotor is to be varied, under different load conditions, while the speed of the servomotor is to follow a certain profile. A tachometer placed at the same shaft would generate a voltage proportional to speed. The supply voltage then can be automatically adjusted so that the speed read by the tachometer voltage follows a reference voltage. The approach is to find the error signal, that is, the difference between the desired and the actual. Here the desired comes

from the outside, and the actual comes from the tachometer. A set of opamp-built circuits are used to generate the output voltage based on the error signal  $\epsilon(t)$ , its derivative  $\frac{d\epsilon(t)}{dt}$ , and its integral  $\int_0^t \epsilon(\tau) d\tau$ . In many applications, the derivative part of the control signal would not be used and only the proportional and integral components would comprise the control signal.

Analog PID control has been the workhorse of industrial control since the early part of the 20th Century. The PID controller was best suitable for continuous processes, such as in petrochemical industries. It is not surprising that the main ideas and development came from chemical engineering. Discrete control at that period was left to programmable logic controllers, usually comprising arrays of relays. As we closed the 20th Century, we see that both approaches have given way to the digital controllers, starting with the mid-century computer numerical control (CNC) machines.

### 3.3 EXPERIMENTATION AND REFLECTION

1. Start at various points in circuit in [Figure 3.2](#) and follow the signals, verifying that the circuit indeed simulates the mechanical system.
2. Study the circuit in [Figure 3.2](#) and assign values to the resistors and capacitors.
3. Build or simulate the circuit in [Figure 3.2](#). Examine if the circuit behaves as expected.
4. There are interesting things you can do with the analog computer that you cannot in the physics lab with real masses and springs. Suppose mass were negative. That is when you accelerate a mass, suppose instead of resisting the acceleration, it aided the acceleration – some sort of negative inertia. How would such a mass suspended from a rubber band behave? First predict its behavior, and then modify the circuit to simulate its behavior. Was your prediction correct? Repeat the thought experiment with negative friction, and then see if your predictions were correct.
5. Pick a few differential equations from your mathematics textbooks and construct circuits that would simulate the dynamics defined by the equations.

*Such thought experiments, predictions, and testing by a model help cultivate engineering intuition.*

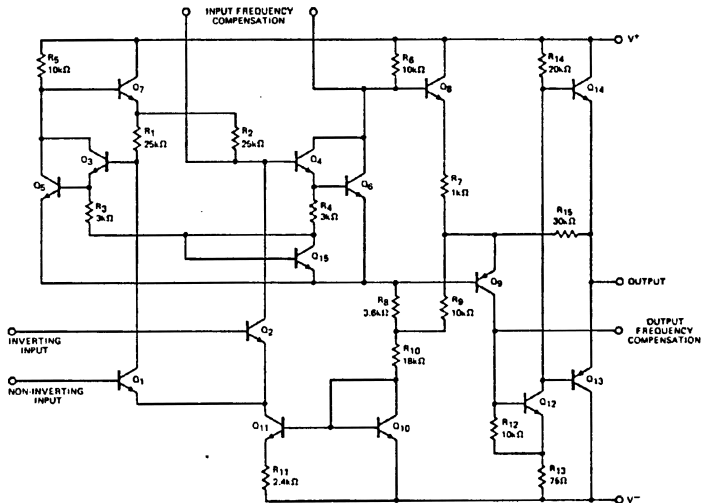
6. Pick a few differential equations from your mathematics textbooks and construct circuits that would simulate the dynamics defined by the equations.
7. Is it possible to construct circuits that would implement non-linear differential equations? If so, give an example.



## Part III

### SIGNAL PROCESSING

Simple signal processing. Simple filters. Watch out – simple things tend to get out of hand quite easily.



Fairchild Linear Integrated Circuits  
 $\mu\text{A} 709$  High-Performance Operational Amplifier, 1965

FREQUENCY AND GAIN

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In typical electrical engineering classes, the topic sooner or later comes to frequency-domain content. You cover frequency and phase responses, and the likes. I often chose to frame the topics around audio signals. Of course, the insights and methodologies will apply to all AC signals, such as telecommunications applications. Typical applications involve oscillators, filters, modulation, decoding, detection, waveform shaping, etc. Our purpose here is not to present a complete treatise of AC signal processing, but rather illustrate how opamps may be used to accomplish some of these functions.

## 4.1 THE KEY INSIGHT

Engineering intuition says that if you put a sinusoidal signal through an opamp running in some linear mode, its frequency should be preserved. We know that the voltage-current relations of capacitors and inductors involve differentiation or integration. The derivative of a sinusoidal function still looks sinusoidal. After all the derivative of  $\sin()$  is  $\cos()$ , and this may even be thought of shifted  $\sin()$ . In technical parlance, this shift is referred to as a *phase difference*. Referring back to the integrator, let us experiment with various parameters and see how the output of the integrator differs from the input. Since both the input and the output are sinusoidal, the difference is in their amplitude and their phase. Recall the basic integrator was given in Section 2.4.

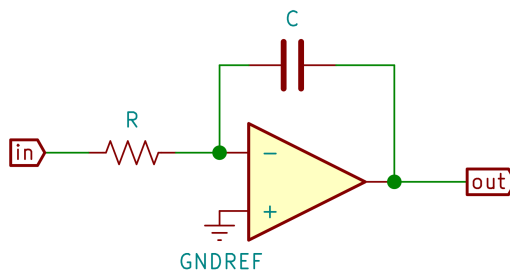


Figure 4.1: Integrator.

If you try different values for  $R$  and  $C$ , you will notice that the output amplitude is inversely proportional to both  $R$  and  $C$ . The integrator looks much like the inverting amplifier, except that the feedback resistor  $R_2$  is replaced by the capacitor  $C$ . The inverting amplifier had a gain of  $\frac{R_2}{R}$ . The gain here is also inversely proportional to  $R$  ( $R_1$  in the amplifier circuit). However, here, the gain is inversely proportional to the (feedback) capacitor. Using the integrator [Figure 4.1](#), with an input of 1V AC, it is clearly visible from [Figure 4.2](#) that as the capacitance doubles, the output amplitude is halved. Note also that sometimes the circuit amplifies the input 1V AC signal, while sometimes it attenuates. Of course, the gain depends on the inverse of the time constant.

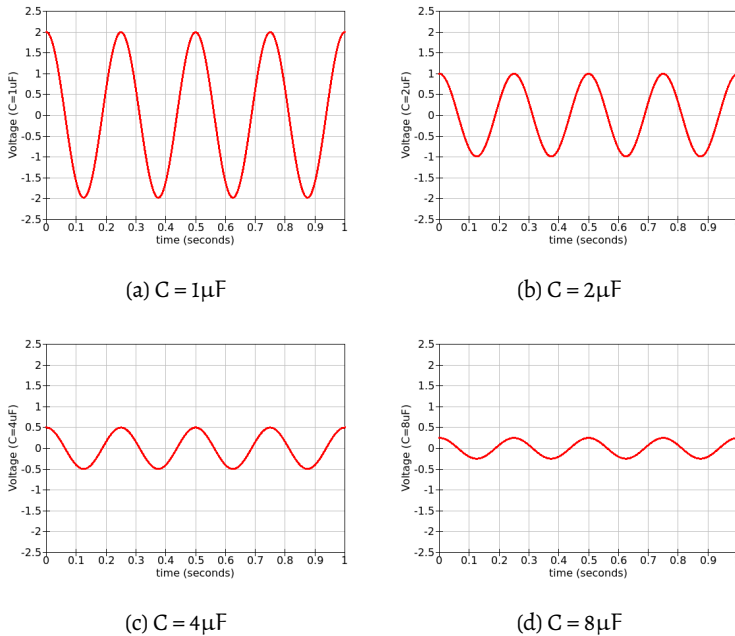


Figure 4.2: The Effect of Capacitance on Gain.

The output of the integrator ([Figure 4.1](#)) when the input is a 1V AC signal.

This makes sense, since the larger the capacitor, the easier it allows AC current to flow, or in other words, the less “resistance” to AC signals. When it comes to AC we rather use the word *impedance*. This explains the term which we identified as the “time constant”  $RC$ . The gain of an integrator is inversely proportional to  $RC$ , as developed in [Section 2.4](#).

$$V_{\text{out}}(t) = -\frac{1}{RC} \int_0^t V_{\text{in}}(\tau) d\tau \quad (4.1)$$

Let us now dwell on the concept of impedance and the implications of signal frequency. Whereas for resistors, for which the resistance is independent of the type of signal, for capacitors, this is not the case. The degree to which a capacitor *impedes* current depends on the type of signal. Clearly, if there is a DC charge on the capacitor, no current flows. At the other extreme, fast changing signals flow through easier than slow changing ones. Current is often likened to water flowing through a pipe. The so-called *hydraulic model* is rather rudimentary, and at times misleading, but here we may make use of it. A capacitor is like a diaphragm placed in the pipe, blocking flow. However, the diaphragm gives a little under pressure. So any vibrations in the water, say sound, will “*pass through*”. A large capacitor is like a flexible diaphragm, while a small capacitor is like a stiff diaphragm.

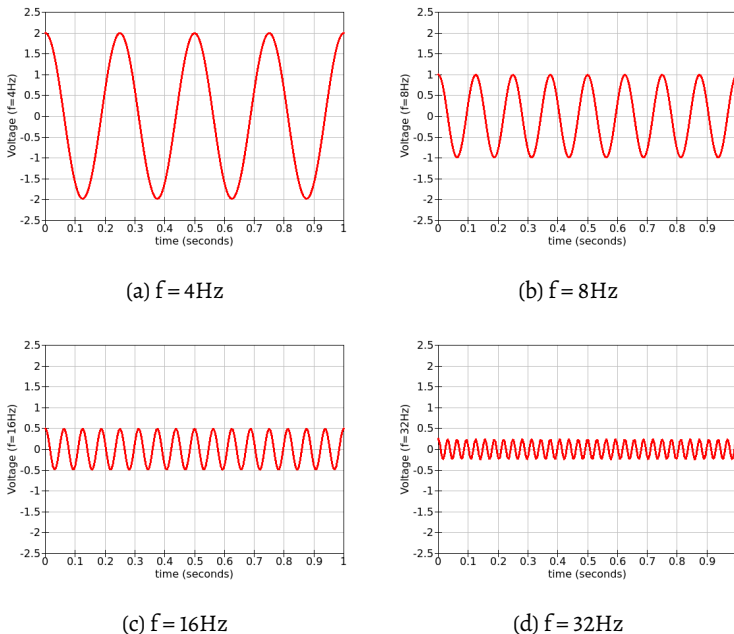


Figure 4.3: The Effect of Frequency on Gain.

The output of the integrator (Figure 4.1) when the input is a 1V AC signal.

Experimentation builds insights which make the theory obvious. Again, Figure 4.3 shows that the output amplitude is halved each time

the frequency is doubled. Once you have a feel that higher frequency will face less impedance through a capacitor, and that the gain will thus be lower, we may review [Equation 4.1](#) and proceed in a formal manner. First, let us express the input signal as a sinusoidal function of time. That is, let us express the voltage at the input as a function of time as,  $V_{in}(t) = A \cos(\omega t)$ . Here  $A$  is the amplitude, and  $\omega$  determines the frequency. We view the argument of  $\cos()$  as an angle. This is just customary jargon from trigonometry, not really much to do with electrons. The  $\cos()$  function is a periodic function. For any integer  $k$ ,  $\cos(x) = \cos(x + k2\pi)$ . At time  $\omega t = 2\pi$ , the  $\cos()$  function completes its cycle and starts over. The length of time it takes to complete a cycle is called the *period* of the signal. It takes  $t = \frac{2\pi}{\omega}$  time units for each period of the  $\sin()$  function. The frequency  $f$  is the reciprocal of the period, that is, how many cycles per time unit we have. We say,  $f = \frac{\omega}{2\pi}$  but more often you see it written as  $\omega = 2\pi f$ .

Now that we have a formal description of  $V_{in}$  we may carry out the integration in [Equation 4.1](#).

$$V_{out}(t) = -\frac{1}{RC} \int_0^t A \cos(2\pi f \tau) d\tau \quad (4.2)$$

$$V_{out}(t) = -\frac{A}{2\pi f RC} \sin(2\pi f t)$$

Since we described the  $V_{in}$  as a  $\cos()$  function, let us also write  $V_{out}$  as a  $\cos()$  function. Using  $\omega = 2\pi f$  reduces clutter, so let us also throw that into the equation. This all is summarized as, when the input is,

$$V_{in}(t) = A \cos(\omega t) \quad (4.3)$$

the output becomes,

$$V_{out}(t) = -\frac{A}{\omega RC} \cos\left(\omega t - \frac{\pi}{2}\right) \quad (4.4)$$

It is not surprising that the amplitude  $A$  retains its place in the output. This is quite intuitive. All things being the same, an input twice the amplitude should indeed generate an output twice the magnitude. Our observations that the gain of the integrator is inversely proportional to both  $R$  and  $C$  (or the time constant) is also visible from the equations. Moreover, the effect of signal frequency, as we had observed,

is clearly visible in the equations. It appears that the gain is inversely proportional to the frequency as well.

There is one final observation. We see that the output lags the input by  $\frac{\pi}{2}$  radians. Besides the mathematical development, there are several insights that lead us to this very observation. Perhaps the easiest way to see this phenomenon of lag is to study the waveforms.

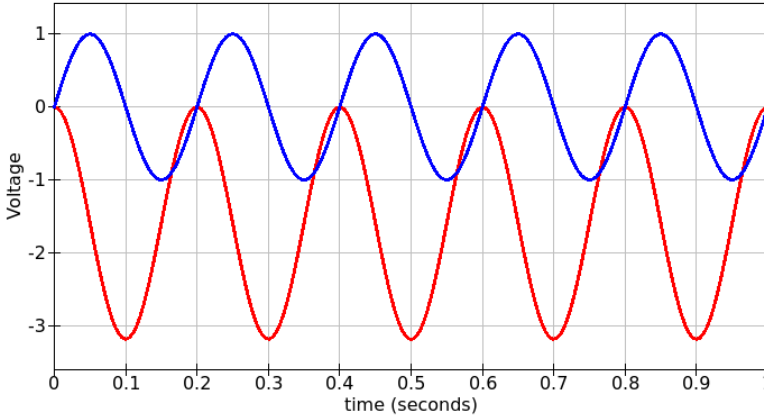


Figure 4.4: Integrator Lag.

The input of the integrator (Figure 4.1) is shown in blue, and the output, in red.

Here the input  $V_{in}$  is in blue and the output  $V_{out}$  is in red. The input has a frequency of 5Hz. Five cycles of the signal is shown, which spans 1 second. Integrating the input (blue) signal, starting at  $t = 0$ , the integral monotonically increases until  $t = 0.1$ . This is the half (upper lobe) of the sinusoidal waveform. At  $t = 0.1$ ,  $\omega = \pi$ . So, the peak of the integral is reached when the input completes half a cycle. Note that, due to the use of the inverting input, the integral has the opposite sign. So, the output (red) line actually reaches its lowest point at  $t = 0.1$ . During the second half of the cycle, the input is negative. So the integral is now reduced, until the cycle completes at  $t = 0.2$ . Since the upper lobe of the input that adds to the integral, and the lower lobe that subtracts, are identical other than their sign, the integral at the completion of the cycle at  $t = 0.2$  is once again zero. We observe this as the output (the red line) is once again at voltage zero at  $t = 0.2$ . Clearly, this relationship leads to a lag of a quarter cycle, and hence the  $\frac{\pi}{2}$  in the argument in Equation 4.4.

## 4.2 INTEGRATOR AS A LOW-PASS FILTER

Since the gain of the integrator is inversely proportional to frequency, it may already be used as a low-pass filter. Plotting the gain as a function of frequency sounds like a sensible way to proceed and gain some insights. The integrator circuit (4.1) was built with a 10k resistor and a 1 $\mu$ F capacitor. The input is a 1Volt sinusoidal signal (the blue line), this way, the amplitude of the output is also the gain of the circuits. We vary the frequency from 10Hz to 1kHz and plot the amplitude of the output (the red line) in Figure 4.5. We may plot using linear or logarithmic coordinates. The figure presents all possible combinations. Which one is easier to read?

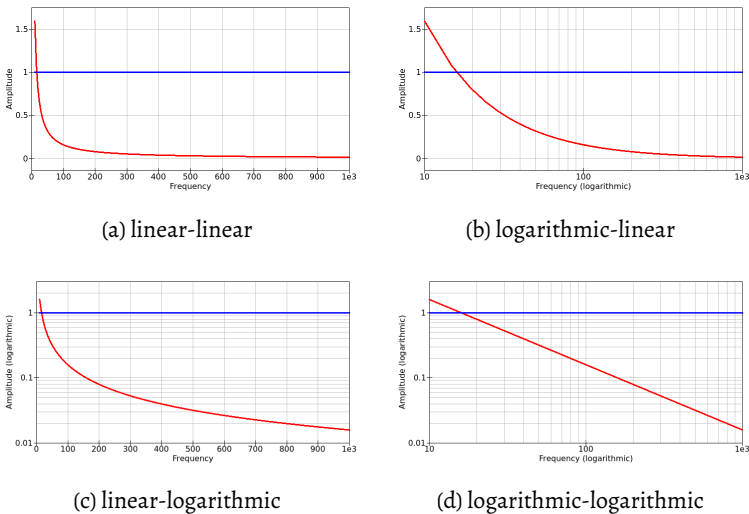


Figure 4.5: The Effect of Frequency on Gain.

Most would say the logarithmic-logarithmic (in engineering, referred as *log-log*) graph is the easiest to read. Why is the log-log graph a straight line? Well, we had seen (Equation 4.4) that the gain is inversely proportional to the frequency. That is, when frequency doubles, gain is halved, and so on. When you multiply by a factor, you increase its logarithm by the logarithm of that factor. The logarithm of its gain will then decrease by the logarithm of the factor. Simply put, multiplication and division appears as addition and subtraction on a log-log graph. The straight line on the log-log graph means that the relationship is a purely reciprocal one.

## 4.3 DECIBEL – A BIT OF HISTORY

All this is fine and dandy, but you usually see gain defined in a unit called *decibel* abbreviated as *dB*. And you see things written like “20 dB/decade”. Sometimes it is “per octave”. What does all that mean?

The history of decibel is indeed an interesting one. It all starts with the new invention of the telephone. As the Bell Telephone Company started connecting customers in the late 19th Century, signal loss over their standard wire became an important engineering consideration. The original unit used was the “*Miles of Standard Cable*” abbreviated as *MSC*. It is the power loss along one mile of the standard double-strand cable (called a loop cable) at approximately 800 Hz (5000 radians per second, to be exact). So, we must be careful to recognize that *MSC* is a power loss unit, not really a unit that describes the amount of cable. The standard was chosen so that a 1 *MSC* of power loss is barely detectable by the human ear.

Here is how I envision what company engineers were facing way back when.

- management: Don't over-design it! Keep costs down, you hear?
- engineers: Look, if we skimp on the cable, there will be some power loss.
- management: Meaning?
- engineers: People will not hear each other clearly.
- management: We don't want that, do we now? Make it so there is no loss.
- engineers: There is no such thing as *no loss*. The heavier the cable the less the loss.
- management: Does “heavier cable” mean “more expensive”?
- engineers: Duh! It is copper. Heavier means more copper.
- management: So, what do we need for a reasonable set-up?
- engineers: Look, the human ear can detect a reduction of such and such. We need to keep under that if you want good sound quality.
- management: Good, make it so. It wasn't that difficult now, was it?
- engineers: Well, if you want the loss to be at the human detection threshold, and you want to connect from coast to coast, you



- will need a cable fatter than a swamp possum with the mumps. Will cost you.
- management: It doesn't need to be coast to coast, don't exaggerate.
  - engineers: How far do you want the line to reach?
  - management: A mile? That's good. Isn't it?
  - engineers: You fellows are the boss. What do we know?
  - management: Sure, a mile. We call that local calls. It will be the default. If they want anything further, we will call it "*long-distance calls*" and charge them an arm and a leg. Geez we managers are smart!
  - engineers: OK. Got it. We design the cable so that over a mile the power loss will be barely detectable.
  - management: Good. We use that cable. What is it called, by the way?
  - engineers: Uuumm, well, actually... it is known as the "*Standard Cable*". Yes, that's it – good old standard cable. That's what you want.
  - management: Wait wait. What happens if we hang a line two miles long?
  - engineers: Well, that would be 2 em-see-ess, won't it?
  - management: What is emm-see-ess?
  - engineers: It is MCS, miles of standard cable. 2 MCS, twice the power loss. 3 MCS, thrice...
  - management: ...and 4 miles, 4 MCS, right?
  - engineers: Riiight. You fellows are sure smart.

The cable that would yield this power loss, as Bell engineers designed it, in modern engineering terms, has 88 Ohms of resistance and 54nF of shunt capacitance per mile. Note the logarithmic nature of the unit MSC: each mile *adds* one MSC of attenuation. For instance, if you have 2 miles of cable, the first mile has an attenuation – the power is reduced by a factor. The second mile also reduces the power by the same ratio, so it becomes the multiplication of ratios. But the "miles" are added. Sounds logarithmic, does it not?

We should pause here and take note of another aspect: the telephone engineer was interested in measuring gain (and loss) of power, not voltage. The graphs we plotted so far (e.g. [Figure 4.5](#)) are all in terms

of voltage gain. The sound level is related to power not voltage. Luckily, we know that power is proportional to the square of voltage. So, we could describe gain in terms of power, not voltage, quite easily. This is a trivial task if we use log-log graphs. Since power is proportional to the square of the voltage, all we need to do is to multiply the logarithm of voltage by 2. The log-log graph in 4.5 would still be linear, but with double the slope if the vertical axis is power rather than voltage. This is yet another advantage of using log-log graphs.

As time went on and telephone connections started connecting countries across national borders, an international standard was sought. The effort yielded what was known as the *Transmission Unit* or TU. Here is how they defined the TU: take the ratio of the output power to the input power. This ratio will be less than 1 for cables, meaning there is a loss of power. Take the logarithm of this ratio. When there is a power loss, the logarithm will be negative. Positive logarithms mean a gain in power. If the logarithm is zero, then there is no loss or gain (often called “*unit gain*”). Due to the capacitance of the cable, the higher the frequency, the greater the loss. The frequency at which the gain is unity is called the *cut-off frequency*. We can compute easily the cut-off frequency of our integrator from Equation 4.4 as

$$f_{\text{cutoff}} = \frac{1}{2\pi RC} \quad (4.5)$$

Now, the TU is defined to be such that it takes 10 TUs to change the logarithm by 1. The TU is very close to the MSC. Bell renamed the TU as a *decibel*. The “*deci*” comes from the 10, and the “*bel*” comes from the name of the inventor Alexander Graham Bell.

This unit of measurement always sounded to me more like a craft definition, rather than an engineering definition. In modern terms, here is how I see it. First, decibel measures a ratio, not an absolute quantity. The ratio of output to input in an opamp circuit, for example, may be measured in decibels. But you may use decibels for any ratio, not just things electrical. You probably have heard of sound volume expressed in decibels. N decibels corresponds to a ratio of  $10^{N/10}$ . If you like, 10 decibels, or “a bel”, corresponds to a ratio of 10. I tend to think of that 10 decibels is an order of magnitude. So, to me, a decibel is like a tenth of an order of magnitude, logarithmically speaking, of course. Multiply 10 of these decibels and you get an order of magnitude.

## 4.4 REVISITING THE INTEGRATOR AS A LOW-PASS FILTER

Now, looking at the response graph (Figure 4.5), we see that for low frequencies, there is gain, at a certain frequency, the gain is one, and then the gain drops, as there is a loss in voltage. The log-log graph is good, because it allows us to work with straight lines. The filter response is a straight line on the log-log graph means, double the frequency, half the gain, also as seen from Equation 4.4. So far so good. But...

Why do engineers simply not say “double the frequency, half the gain”, but instead say things like “-6dB per octave” when referring to these graphs. Well, when it comes to frequency, “double the frequency” has a deeply rooted tradition from music. It is called an octave. This, however, does not carry over to other ratios – here the ratio of the output voltage to the input voltage. The latter, as we saw, is measured in ten steps to an order of magnitude. Perhaps it is also a source of confusion that an order of magnitude is also 10. The 10 steps and the order of magnitude are somehow related by nature, but not by physics. Let me explain. . There is no reason we must divide the order of magnitude into 10 steps. On a guitar, the octave is divided into 12 frets, and we are fine with that. The quantity 10 pops up all over the place because we count in tens, because we have ten fingers. If we had eight, our order of magnitude would most likely be eight, and we would most likely be using an octal numbering system. Better if we had 16 fingers. Then computer engineers would be right at home using hexadecimal numbers.

*Although in early computer days the Latin term sexagesimal was used, it was renamed as hexadecimal by IBM, who preferred the abbreviation hex over the alternative.*

So, although we have a log-log graph, and we count ratios of frequencies on the horizontal axis and ratios of frequencies on the vertical axis, we use octaves for frequency and decibels for voltage. Could it have been easier – I suppose so. But as you see, there is quite a bit of history behind the use of terminology for each of the axis.

So, what is the slope of the log-log graph? Well, half voltage for double the frequency, or half the voltage per octave, or  $\log(0.5) = -0.3010299957$  bels per octave, or  $10\log(0.5) = -3.010299957$  decibels per octave. If you think it is over, well, no. As Yogi Berra said, “it isn’t over until it’s over”. First, it is usually written that the slope of the log-log response graph of the low-pass filter is -3dB per octave, not -3.0103 dB. This is a close approximation within three parts in a thousand. Next, in some context, you see that it is -6dB per octave. This

is also correct. The  $-3\text{dB}$  refers to the voltage gain, whereas the  $-6\text{dB}$ , with double the slope, refers to the power gain.

Finally, we have come to the end of our story. The integrator opamp (Figure 2.4) has a power gain of  $-6\text{dB}$  per octave. Double the frequency, and you get a  $-6\text{dB}$  power gain (or a  $6\text{dB}$  power loss). If you do not favor expressing the horizontal axis in octaves, there is another common convention. Whereas the octave refers to doubling the frequency, the term *decade* is used to describe a ten-fold increase in frequency. So, the voltage gain here would be  $-10\text{dB/decade}$ , or equivalently, the power gain would be  $-20\text{dB/decade}$ .

It is common that this filter will be referred to as a *first-order low-pass filter*, since it contains only one capacitor, and hence, the input-output characteristics of the filter is described by a first-order differential equation.

#### 4.5 A PRACTICAL LOW-PASS FILTER

In typical applications, the basic circuit (4.4) is modified as given in Figure 4.6. Let us motivate the idea behind the feedback resistor  $R_2$ . When the signal frequency is lower than the cut-off frequency, the capacitor will have a high enough impedance that the circuit behavior will be determined mostly by  $R_2$ . Here  $R_1$  and  $R_2$  form a plain old inverting amplifier with a gain of  $\frac{R_2}{R_1}$ . When the frequency is higher than the cut-off frequency, the *path of less resistance* is through the capacitor, and thus, high frequency signals are attenuated. The net effect of the feedback resistor can easily be depicted from the frequency response of the circuit.

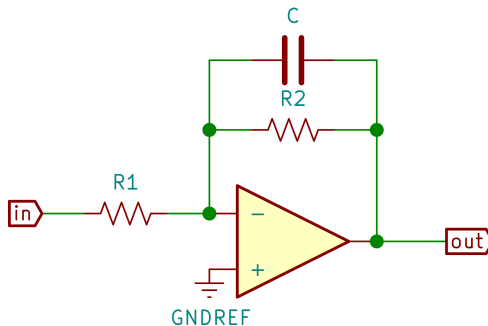


Figure 4.6: Low-Pass Filter.

Let us pick some component values. Let us set the cut-off frequency to 1 kHz, and a low frequency gain of 2. Now we can compute  $R_1$ , for a capacitor of, say, 10 nF. The gain is 1 at the cut-off frequency, so  $2\pi \cdot 1000 \cdot R_1 \cdot 10^{-8} = 1$ . This gives us  $R_1 = 15916 \Omega$ . We pick a standard 15 k resistor from our parts box as a good approximation. At low frequencies, the gain is 2, so  $\frac{R_2}{1k} = 2$ . A standard 33 k resistor would work nicely enough. If you are going to be an engineer, I guess you have to learn not to nitpick.

*If you are comfortable with a 15k resistor to be close enough, congratulations. You sure have some engineering pragmatism.*

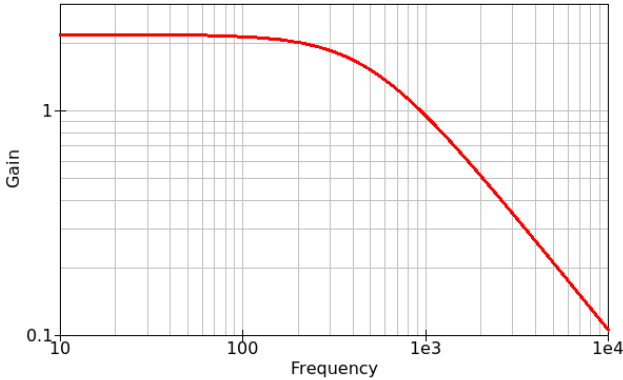


Figure 4.7: Low-Pass Filter Frequency Response.

Looking at the graph (4.7) we see that the gain at 1 kHz is just a bit below unity. This is what we aimed for. The gain starts at a little over 2 (10% over to be exact. The gain is  $\frac{33}{15}$ ). That too is according to plan. The gain indeed drops 10 dB/decade as it is set by the nature of the first-order filter. At 1 kHz, the gain is 1, while at 10 kHz, it is 0.1.

#### 4.6 ACTIVE VERSUS PASSIVE FILTERS

Here is a legitimate question we did not address: why can we not use a simple resistor-capacitor filter? The answer is that such a filter will always have a gain less than 1. Only active filters may have a gain greater than 1. What if we fed the output of the passive filter to an opamp amplifier, would that not work. Well, in theory it would. But it is best to prevent signal loss as much as possible. The active filter built around the opamp uses the capacitor-resistor combination as an integral part of its operation. Whereas a resistor-capacitor filter would cause a loss in signal power. Another way to see this is that the input impedance of

the active filter is many orders of magnitude greater than the resistor-capacitor filter. The difference becomes important as you start dealing with noise, especially when dealing with delicate signals that have very little power. Would you rather feed the signal from a radio-telescope to a resistor-capacitor network, or an active filter?

#### 4.7 EXPERIMENTATION AND REFLECTION

1. Build the circuits in this chapter and verify the observations made here.
2. Use an audio source, cascade several low-pass filters and see what the output sounds like. How does the sound change as you keep adding filter stages?
3. How would you build an active high-pass filter? Give several approaches. Guess what the advantages and disadvantages of each approach would be.

## HIGHER-ORDER FILTERS

## 5.1 NOT STEEP ENOUGH?

An *ideal* low-pass filter should be all-or-nothing. It should have constant gain until some frequency, and then have no output whatsoever. A  $-6\text{dB}$  per octave filter may sound far from ideal to a purist. The fact is, it really depends on the application. However, there is a simple way you could push your low-pass closer to the ideal. Just cascade two filters, one after the other. Remember the MSC? If you put one  $-6\text{dB}$  filter after the other, you should get a  $-12\text{dB/octave}$  filter, right?

Using the same values as in the previous section (4.5) we once again run the circuit and observe its frequency response. It is best to do this in the lab using real opamps. The next best thing is to simulate it.

But wait a minute. The low-frequency gain in Figure 4.6 was set to 2, and we even had a 10% extra from not using the exact resistor values. If we cascade two identical stages, would we not get a gain of 4+? Yes, indeed. So, what to do? Well, looking at Figure 4.6, we should not change  $R_1$ , because  $R_1$  and  $C$  determine the cut-off frequency. We may change  $R_2$ . Let us leave one of the stages exactly as before with  $R_2 = 33\text{k}$ , but make  $R_2 = 15\text{k}$  at the other stage. That way, the gain at low frequencies would be unity. The gain at low frequency of the tandem filters would be set by one stage, and the other would have unit gain.

Sounds fair. Let us try.

Looking at the graph (Figure 5.1), we see that low-frequency gain is kept around 2 as desired. The gain now drops  $20\text{ dB/decade}$ . At  $1\text{ kHz}$ , the gain is 1, while at  $10\text{ kHz}$ , it is 0.01.

## 5.2 COOKBOOK RECIPES

Many times seasoned engineers would resort to a few *cookbook* configurations. These are the bread and butter of electronics professionals. They are tried and tested configurations handed down from genera-

*Benjamin Brewster wrote in The Yale Literary Magazine of February 1882: "I heard no more, for I was lost in self-reproach that I had been the victim of "vulgar error." But afterwards, a kind of haunting doubt came over me. What does his lucid explanation amount to but this, that in theory there is no difference between theory and practice, while in practice there is?"*

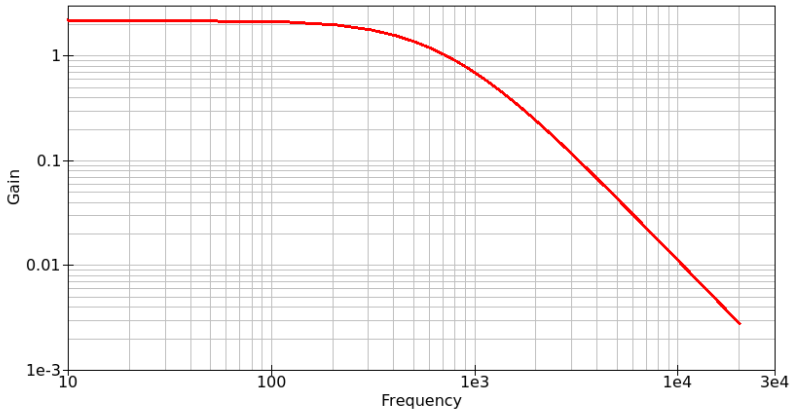


Figure 5.1: Low-Pass Filter Frequency Response of Two Cascaded Filters.  
(Based on the circuit given in [Figure 4.6](#))

tions – the anonymous folk songs and mythologies of filters past. You must have these in your repertoire if you want to be recognized by your fellow craftspersons.

Here is what most would do when asked for a second-order low-pass filter ([Figure 5.2](#)).

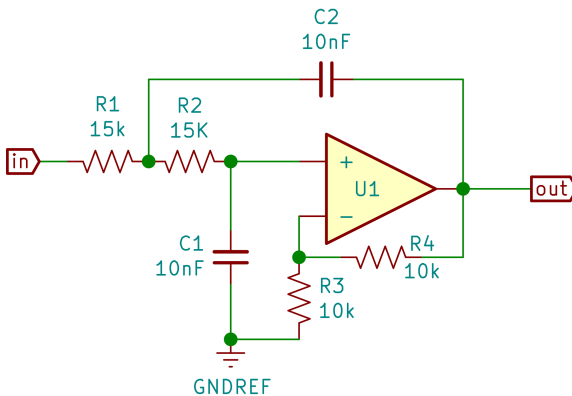


Figure 5.2: Second-Order Low-Pass Filter.

Strictly speaking, the two resistors and the two capacitors that make up the filter need not be identical. Making them identical however simplifies the computations. In production, it reduces assembly errors and streamlines inventory keeping. It may also give you a sense of balance. The filter has a low-frequency gain of 2. Its frequency response



shows the characteristic  $-20\text{dB}/\text{decade}$  ( $-6\text{dB}/\text{octave}$ ) gain after the cut-off frequency of  $1\text{kHz}$ .

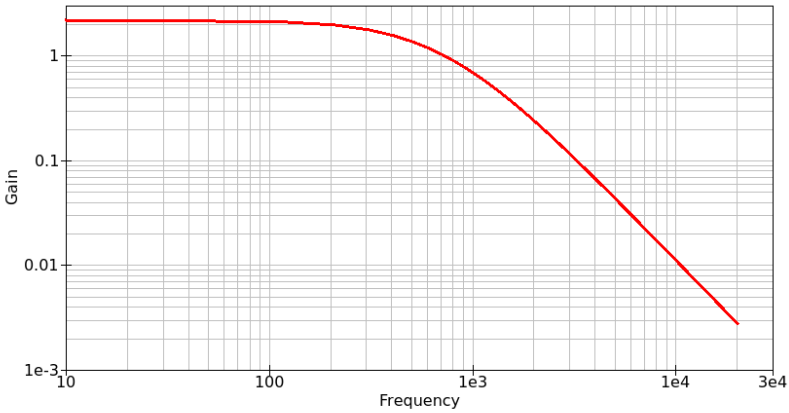


Figure 5.3: Second-Order Low-Pass Filter Frequency Response.  
(Circuit given in [Figure 5.2](#))

If you want to make a high-pass filter, you may start with the circuit given in [Figure 5.2](#) and simply exchange the places of the resistors and the capacitors, as depicted in [Figure 5.4](#). There seems to be some symmetry. If this appeals to your sense of beauty, once again, congratulations, you have the makings of a good engineer.

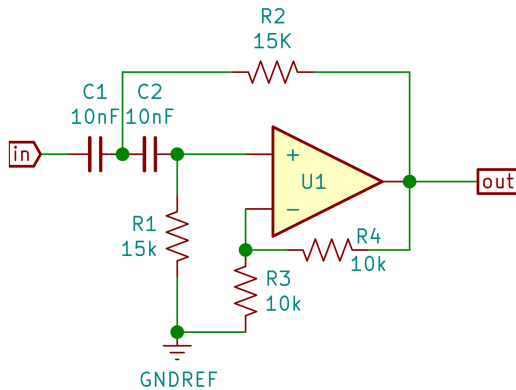


Figure 5.4: Second-Order High-Pass Filter.

The filter has a high-frequency gain of 2. Its frequency response shows the characteristic  $-20\text{dB/decade}$  ( $-6\text{dB/octave}$ ) gain before the cut-off frequency of  $1\text{kHz}$ .

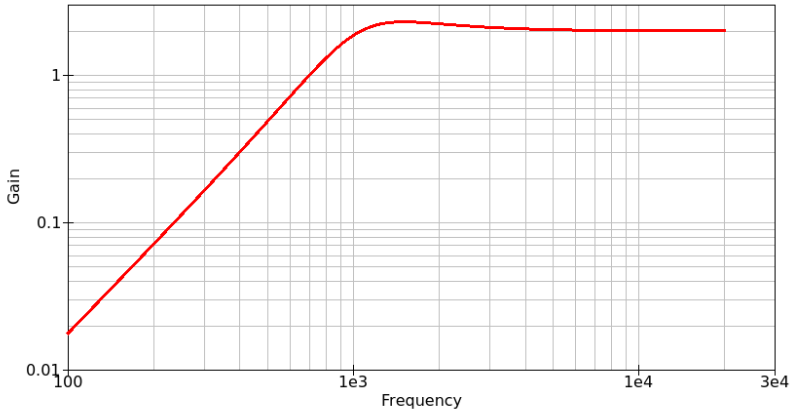


Figure 5.5: Second-Order High-Pass Filter Frequency Response.  
(Circuit given in [Figure 5.4](#))

### 5.3 BAND-PASS AND BAND-REJECT FILTERS

Now that we have the workings of filters, we may cascade them to allow or reject a band of frequencies. In the case of band-pass, first place a high-pass filter with a cut-off frequency  $F_{lo}$  equal to the lower limit of the band. Its output would be the frequencies above  $F_{lo}$ . Now, if we feed this output to a low-pass filter whose cut-off frequency  $F_{hi}$  is set to the upper end of the band, all frequencies above  $F_{hi}$  will also be attenuated. What is left are the frequencies in the range  $F_{lo}$  to  $F_{hi}$ .

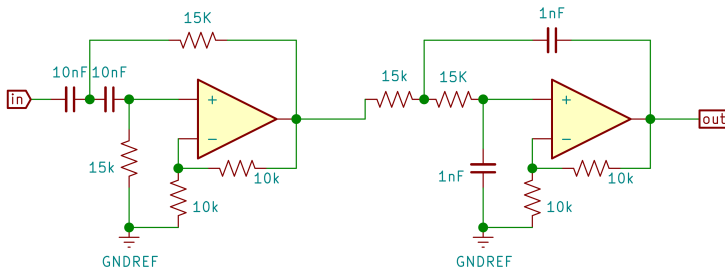


Figure 5.6: Band-Pass Filter.

You may have noticed that we changed the capacitor values of the second (downstream) filter in [Figure 5.6](#) from 10nF to 1nF. This pushes the cut-off frequency of the low-pass filter to 10kHz. This way, the filter passes frequencies in the band 1kHz to 10kHz. The frequency response ([5.7](#)) is as expected – with -20dB/decade (-6dB/octave) slopes around a band from 1kHz to 10kHz.

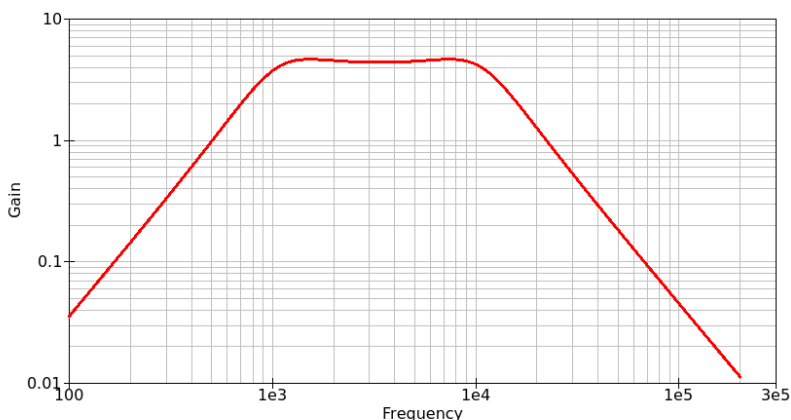


Figure 5.7: Band-Pass Filter Frequency Response.  
(Circuit given in [Figure 5.6](#))

Now, suppose we wanted the opposite – to reject a band of frequencies and allow all frequencies below or above a band. This is called a *band-reject* filter. In this case we cannot simply cascade low-pass and high-pass filters. Why not? Well, because there is no frequency that will pass through **both** filters. So, a serial arrangement is not the solution. What we can do is depicted in [Figure 5.8](#).

We take the input to parallel filters, and then once again, combine their output. Essentially, frequencies below the lower end of the band and frequencies above the upper end of the band are united once again at a third summing amplifier stage. Comparing [Figure 5.8](#) to the previous [Figure 5.6](#), do you see anything noteworthy? Yes, the cut-off frequencies are now interchanged. Now, the high-pass filter has a higher cut-off frequency than the low-pass filter. While the filter resistors are still all 10k, the capacitor values are swapped. The 1nF capacitors are now used by the high-pass stage, while the 10nF capacitors are with the low-pass stage. The frequency response is as expected, with a range of frequencies attenuated. The green and blue lines show the gains of the high-pass and the low-pass filters. Unlike the band-pass filter, where cascading the two filters resulted in multiplying their gains for

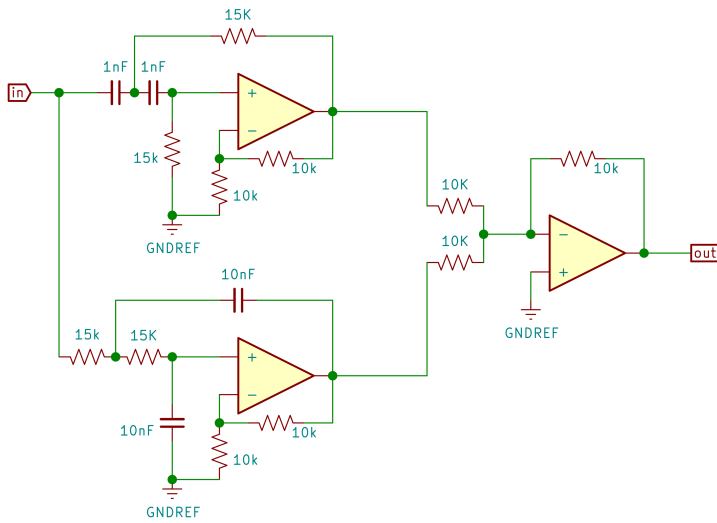


Figure 5.8: Band-Reject Filter.

the band, here the gain outside the band remains the same (a little over 2, remember?).

The band-reject filter is also known as a *band-stop filter*, and by another rather descriptive name. Looking at Figure 5.9, there would be little surprise if one would call this a *notch-filter*, although this name is more appropriate when the filter has a narrow stop band, like the one in the figure.

#### 5.4 SOME GENERALIZATION AND FORMALISM

So, we have seen second order filters (5.2 and 5.4) which seem to share many properties. One is the dual of the other – swapping the places of capacitors and resistors. This configuration is known as the *Sallen-Key* topology.

In both cases, the base gain is 2, which is due to the feedback resistor to the inverting input and the resistor from that input to ground. Note that, in cases where these resistors have the same value, the gain was  $1 + \frac{R}{R} = 2$ . The gain could be reduced to unity if the opamp is configured like a voltage follower. That is. connect the output to the inverting input,

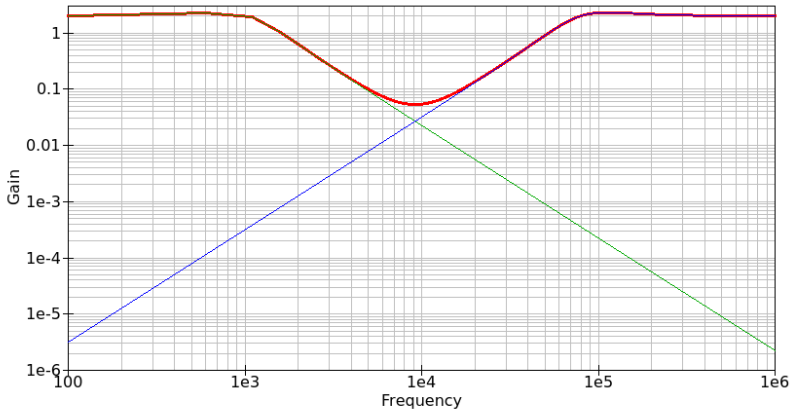


Figure 5.9: Band-Reject Filter Frequency Response.  
(green: low-pass, blue: high-pass, red: combined band-reject)

and remove the resistor from the inverting input to ground, essentially making the gain  $1 + \frac{0}{\infty} = 1$ .

Practicing engineers become familiar with these topologies. Chip manufacturers write application notes, and even put on-line interactive calculators on their webpages where you may input the desired characteristics and the program gives you component values of almost any type of filter.

If you are interested in how these filters work, then we need a bit more mathematics. The differential and integral equations we saw in [Chapter 3](#) are fine, but rather clumsy when it comes to practical derivation. Actually, the brute-force calculus approach is a bit like a sledge hammer. It works but there are shortcuts. The motivation is this: the circuits we have are in some sense, simple linear circuits. They preserve the frequency of the signals. Anywhere to poke, you will see a sinusoidal signal with the same frequency. If we used non-linear components, that would not necessarily be the case. Say, for example, you feed AC power to a full-bridge rectifier.

The blue line ([Figure 5.10](#)) shows the input voltage, and the red, the output voltage. The supply is 10V AC, whereas the rectified power falls short of the 10V, since there is a voltage drop across the rectifying diodes. Nonetheless, looking at the red line, you realize that the frequency is now double the input signal frequency. Plus, the output is not a pure sinusoidal signal – it has sharper downward pointing peaks. This is why we said a diode circuit may not necessarily be a

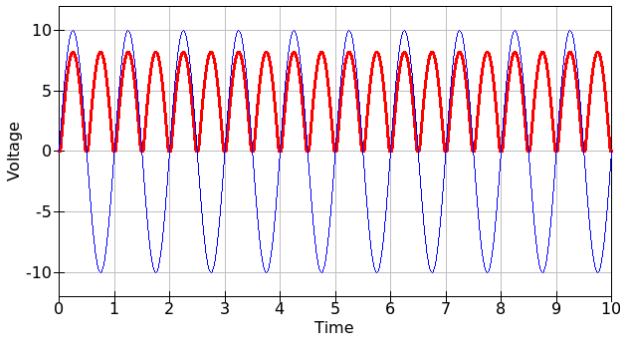


Figure 5.10: Full Bridge Rectifier.  
(blue: input, red: full-bridge rectifier output)

linear circuit. Most circuit you build with resistors, capacitors, and inductors will preserve the frequency. So, how exactly does this help us? Well, a lot of the calculus could be simplified since we need not concern ourselves with the frequency any further than saying it is a constant parameter throughout the entire circuit. The only difference of voltages (and currents) at various nodes of our circuit will differ only by amplitude (magnitude) or phase (lagging or leading). We will revisit this topic in the next chapter. However, before we do so, let us present one more filter topology.

## 5.5 A BIT OF MAGIC

Hopefully, looking at, say [Figure 5.2](#), you may have an intuitive feel for what is happening. The circuit favors low frequencies. First, high frequencies are grounded through C1, and the low frequencies are carried over th the opamp. Same with the capacitor C2, which also sends high frequencies directly to the output. If you look at this circuit and think you may not need a cookbook, that is quite understandable. Bet there are circuits that look quite magical to the untrained eye. The so-called *state-variable filter* is a good example. It is also known as the *Kerwin-Huelsman-Newcomb (KHN) biquad filter*. Other than the Sallen-Key topology, perhaps the most used one is the family of state-variable filter topologies.

It seems, as its name implies, it took three people to come up with this one. But it is quite worth it, as the circuit does the job of a low-pass, a high-pass, and a band-pass filter all at once – and with only

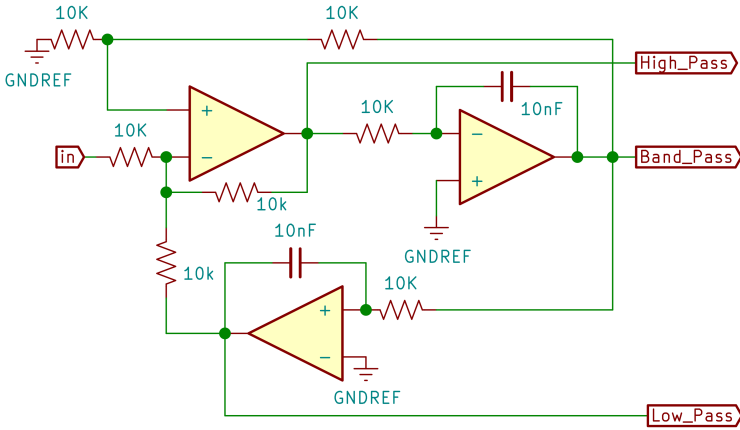
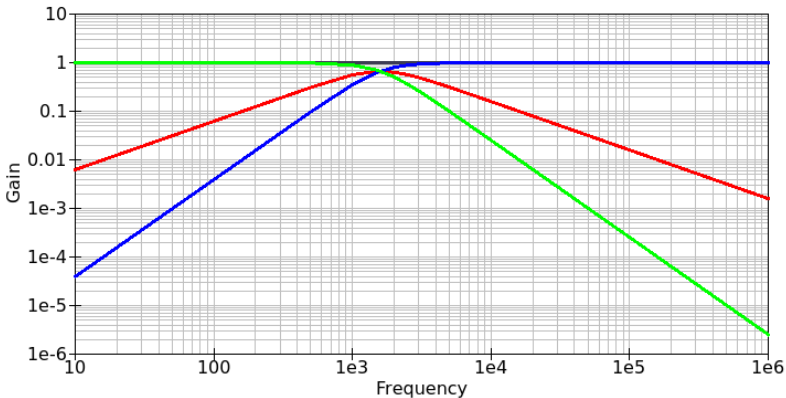


Figure 5.11: State Variable Filter Circuit.

three opamps. The mathematical demonstration of this filter is quite elegant, but outside the scope of this elementary book.

Figure 5.12: State-Variable Filter Frequency Response.  
(green: low-pass, blue: high-pass, red: band-pass)

The frequency response shows that this is indeed a second-order filter, as the filters display  $-20\text{dB/decade}$  ( $-6\text{dB/octave}$ ) attenuation. The green line is the output of the low-pass output, and the blue line, the high-pass output. The red line is the output of the band-pass output, as shown in the preceding circuit.

## 5.6 EXPERIMENTATION AND REFLECTION

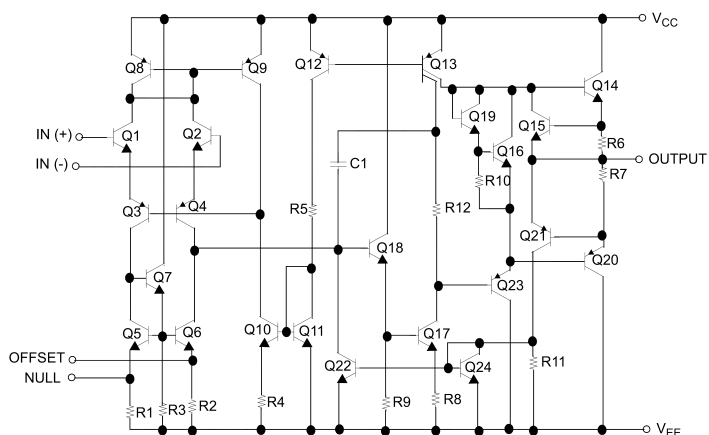
1. Build the circuits in this chapter and verify the observations made here.
2. Use an audio source, compare the cascaded second-order filter output to the ones given here. Which sounds better? Could you venture a guess as to why that might be?
3. How would you build an active filter that passes several bands and rejects some others? Using more and more of the *cascade* and *parallel* configurations might work, but would the design not become a bit unwieldy as more circuitry is involved? Is there a better way?



## Part IV

### FORMALISMS

A more formal mathematical view. It is good to know that we know enough to prove. VFR when possible but IFR when needed.



Fairchild Semiconductor  
 $\mu\text{A} 741$  Single Operational Amplifier, 1968

## PHASORS

Do you want to go a bit deeper into the rabbit hole? Do you want to see why the filters work the way they do? Do you want to understand enough not just to use cookbook recipes, but to design your own? If not, you may skip this chapter. But if you do, stick with me. It is not as bad as it looks.

## 6.1 HINTS FOR A MATHEMATICAL FORMALISM

Life is pretty easy when you work with DC and resistive networks. After all, for any given resistor, the voltage drop  $V$  across the resistor and the current through  $I$  obey the simple relationship  $V = IR$ . Luckily, this relationship holds for the resistor even when the signals are time-varying. That is,  $V(t) = I(t)R$ . If we tried the same with capacitors and inductors, we have to resort to differential or integral equations. Take [Equation 2.8](#) we had for the capacitor we used in [Chapter 2](#).

$$i(t) = C \frac{dV(t)}{dt} \quad (6.1)$$

This differential equation quite nicely describes the behavior of the capacitor. Here is a plot that shows the voltage across and the current through a capacitor.

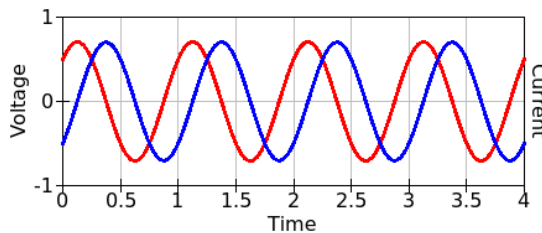


Figure 6.1: The Voltage-Current Behavior of a Capacitor.  
(blue: voltage, red: current)

The blue line is the voltage, and the red, the current. Notice how current lags behind the voltage by a quarter of a period. Another way to see this is to notice that the current peaks when the voltage is increasing the fastest – when its slope is the steepest. This once again conforms to the relationship given by [Equation 6.1](#).

Would it not be nice to be able to write [Equation 6.1](#) in a simpler manner, similar to  $V = IR$  we have for the resistor. Why can we not do that? Well, for one thing, with DC, the signals are not time-varying – they are constant. So, when there is no *function of time* concern, there is no differentiation with respect to time to speak of. With sinusoidal signals, things change over time, so we need beyond simple linear equations, differential equations to express changes in time.

But not all is so desperate.

True, you need differential equations if you have signals that follow arbitrary time-varying functions, but for sinusoidals? Sinusoidals have so much *regularity*. Moreover, the frequency of all signals in a simple circuit made with linear components will be the same. That is, the signal may be varying, but the frequency is constant. You may describe them not with a single constant like in DC, but with only two quantities beyond the circuit-wide invariant frequency. A sinusoidal function is uniquely determined by its frequency, its amplitude, and its phase. Since our well-behaved linear circuits preserve the frequency, we only see different amplitudes and phases around our circuit. So, while we may not write scalar equations such as  $V = IR$ , we may write equations that embed amplitude and phase. An algebra that works with vectors of size two would be sufficient.

*If this sounds like circular logic, well, it is. The so-called linear components are those that allow the circuit to be described by linear (differential) equations.*

Here are the desiderata: we would like to easily add, subtract, multiply, divide, differentiate, and integrate using our variables. Rather than a full-fledged differential-equations formalism, what can we use to take advantage of the simplifications presented by only looking at sinusoidal functions of the same frequency? This is not a question unique to electricity. In physics, many phenomena are described by waves. Physicists use *phasors* and *phasor algebra* in such cases. Typical in many cases, the way the concept emerged, and how it is now presented differ quite a bit. The phasor is no exception. Often, there are insights and experimental results that yield useful viewpoints. Then come a formalism that makes things succinct by providing definitions, axioms, rules, etc. Almost all textbooks will take this route. Here we will try, at least to some extent, to motivate the ideas. If you would like

to build an understanding of the historical development of how the concept of phasors came about, take a look at the excellent exposition by Araújo and Tonidandel [1].

## 6.2 A NICE WAY TO VISUALIZE PHASORS

The oscilloscope, or simply *scope* among seasoned engineers, is how we observe and *visualize* the dynamics of time-varying signals. When you view a sinusoidal wave on the scope it goes up and down in the customary manner. If you turn off the sweep, the sinusoidal will simply be a point swinging up and down, almost like a pendulum. It slows down towards the ends, and gains speed in the middle. Now imagine a clock with only its minute hand, viewed from the edge. Would the tip of the minute hand not seem the same? This is the key idea of phasors. Surprised? Wait wait. Suppose you have a bunch of clocks, some big some small, strewn randomly around the floor. Let the top 12 o'clock mark on the clocks face all different random directions. Now each, when viewed from the edge will look like our no-sweep scope signal. Note, however, that all our clocks share one thing in common. They complete their tour around the clock face in exactly 60 minutes. That is, they all have the same period, or the same frequency. If you take any two clocks, the relative angle between their arms will remain the same. Why? Well, because, they have the same frequency. Now some clocks have long arms and some short. If you wanted to find the position of the tip of one clock's arm from another, you would need two things: how much longer (or shorter) is one arm compared to the other, and what is the (constant) angle between the arms. Now, does this not sound exactly like complex numbers? If you multiply two complex numbers, the result has a magnitude equal to the product of the magnitudes, and the angle is the sum of the angles. The clocks arms may be rotating, but if you consider the arms to be complex numbers on the complex clock-face, then the ratio of one to another is a constant.

*It always amazes me to see cheap sci-fi movies show a sinusoidal signal on a few scopes with a bunch of mesmerized top scientist crowding to observe. What a joke.*

This is what happens in linear circuits. All signals, voltages and currents are sinusoidal functions with the same frequency. They change over time, sure, but their ratios are constant complex numbers as long as we view our sinusoidal functions *edge-wise* or as *phasors*. You may say a phasor is just an edgewise view of a rotating vector.

## 6.3 EULER'S IDENTITY TO THE RESCUE

A good formal way to view phasors is by recalling *Euler's identity*. The phasor here, by the way, is not the *phaser* Captain Kirk uses. The word comes from *phase vector*. We are only up one notch from scalars – from vectors of size 1 to vectors of size 2. Rather than inventing a new vector-2 algebra, phasor algebra uses complex numbers. After all, complex numbers may be viewed as vectors of size 2 (*2-vectors*). They have two independent parts: a real part and an imaginary part. Moreover, they are easy to add and subtract in regular (rectangular form), that is, when written as  $a + jb$ . Here  $j$  is the imaginary unit element of the group of complex numbers, often described as the quantity whose square is  $-1$ . Although this somewhat describes the imaginary unit, strictly speaking, it is better to view  $j$  as an independent dimension that has to do with the phase of signals. Note that electrical engineers routinely use the letter  $j$  rather than the letter  $i$  seen in mathematics books. That is because electrical engineers like to reserve the letter  $i$  for current.

*James Tiberius Kirk, born in Riverside, Iowa, on March 22, 2233, was the captain of the Constitution Class starship USS Enterprise (NCC-1701).*

When it comes to multiplication and division, complex numbers are quite nimble in that as well, provided that you write them in polar form, as  $Ae^{j\theta}$  where  $A$  is called the *magnitude* and  $\theta$  is called the *argument*. The magnitude is also referred to as the *modulus*. Phasors use the magnitude to describe the amplitude of the sinusoidal function, and the argument to depict the phase. With this arrangement, complex calculus fits our needs perfectly. Electrical engineers quite frequently translate between the rectangular form and the polar form, depending on whether addition or subtraction is needed, or multiplication or division.

*The choice of the letter i for current comes from Monseigneur André-Marie Ampère (1775-1836), whose name is used for the unit of current. Ampère calls it "an intensity of current," thus the letter i.*

Wait a minute, you might ask. How can a complex number, or a 2-vector, be the same as a time-varying function? Here is a key concept. Leonhard Euler (1707 - 1783) says that a complex number in polar form can be written as

$$A e^{j\theta} = A [\cos(\theta) + j \sin(\theta)] \quad (6.2)$$

First, we see that it is rather trivial to accommodate the amplitude  $A$ , as it is nothing but a linear scale factor. We may drop it from further discussion until needed. Now, if  $\theta$  is not a number but a function of time, say  $\omega t$  we would have

$$e^{j\omega t} = [\cos(\omega t) + j \sin(\omega t)] \quad (6.3)$$

Here  $\omega$  is our familiar expression equal to  $2\pi f$ . Why? Because when  $t$  is equal to the period,  $\omega \frac{1}{f} = 2\pi$ . That is why. All this means is that we could carry out our calculations using the polar form, and when needed, recover the original time-dependent rectangular.

$$\cos(\omega t) = \Re(e^{j\omega t}) \quad (6.4)$$

where  $\Re(\cdot)$  means the real part of the expression. Alternatively, we may use the complex conjugate of the polar expression to accomplish the same extraction.

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t}) \quad (6.5)$$

Note that phase differences are just as easily accommodated. Say a signal is  $\phi$  radians lagging. Then, the polar form would be

$$e^{j(\omega t - \phi)} \quad (6.6)$$

When the frequency is taken as a known circuit-wide invariant parameter, the polar form of the complex function is often written in the so-called *phasor notation* as,

$$Ae^{j(\omega t + \phi)} = A/\underline{\phi} \quad (6.7)$$

The upshot of all this is that we have a nice tool that allows us to handle circuits subject to sinusoidal excitations, made of capacitors and inductors, almost with the ease of dealing with DC and purely resistive networks. We will not pursue a formal proof of our methods, but in the spirit of our book, try to motivate and provide insights. Hopefully, by the time we see enough examples, and discuss intuitively why things work the way they do, we will have confidence that, as the bread-and-butter of electrical engineering methods, phasor algebra is just dandy.

#### 6.4 THE IMPEDANCE OF A CAPACITOR

Let us put phasor algebra to a test and compute the impedance of a capacitor. After all, a practical application of phasors in polar form is in their use to describe impedance. We will use the complex number description of voltage and current, knowing that we can always translate back. We have,

$$V(t) = ve^{j(\omega t)} \quad (6.8)$$

From the definition of the capacitor, we have, as seen in [Equation 2.8](#),

$$I(t) = C \frac{dV(t)}{dt} = C \frac{d(ve^{j(\omega t)})}{dt} \quad (6.9)$$

Luckily, differentiation is also straightforward when the phasor is in polar form.

$$I(t) = C \frac{dV(t)}{dt} = Cvj\omega e^{j(\omega t)} \quad (6.10)$$

where  $v$  is the magnitude of the voltage. As an analog to resistance, let us define the impedance as  $V(t)/I(t)$ . Then, the phasor  $Z$  is the impedance of the capacitor where,

$$Z = \frac{ve^{j(\omega t)}}{Cvj\omega e^{j(\omega t)}} \quad (6.11)$$

Many terms conveniently cancel, leaving us the impedance

$$Z = \frac{1}{j\omega C} \quad (6.12)$$

Let us pause here and reflect upon the result [6.12](#). First,  $Z$  is a constant, and not a function of time ( $t$ ). Next, it is inversely proportional to  $C$ , which we have been observing all along. If we replace the parameter  $\omega$  by its equivalent  $2\pi f$  we have the alternate definition,

$$Z = \frac{1}{j2\pi f C} = \frac{-j}{2\pi f C} \quad (6.13)$$

We also see that the impedance of a capacitor is inversely proportional to the frequency of the signal. We have observed this too. Most importantly, provided that we do not mind working with complex numbers, we can use impedance in place of resistance in Kirchoff's Law and in Ohm's Law when dealing with capacitors.<sup>1</sup> For instance, if I want to find the current through a capacitor, all I need is to divide the voltage by the impedance of the capacitor – just like when dealing with resistors.

A similar development will show that the impedance of a resistor is the same as its resistance. Likewise, the impedance of an inductor is  $j\omega L$ , where  $L$  is the inductance.

<sup>1</sup> Granted, this statement needs further development if a rigorous proof is sought. But for our purposes, it may suffice, since it strongly appeals to the intuition we have so far developed.

## 6.5 DIFFERENTIATION AND INTEGRATION IN PHASOR ALGEBRA

Before we leave these few sections, let us once again dwell on how easy differentiation was in polar form. Essentially, if you differentiate a polar phasor, you multiply by  $j\omega$ . What about integration? Well, you guessed it. Divide by the same. In all fairness, this is a most elegant field of applied mathematics. I wish we had more time to spend on this subject. There are alternative ways to look at the development here. If you are interested to go deeper into the rabbit hole, you must study Laplace transforms. We will give a brief introduction into the subject in [Section 7.6](#) after we develop a little more intuition. There, differential equations are replaced by their equivalent algebraic equations. The commonly used parameter  $s$  serves the same function as the  $j\omega$  here. This is not a fluke, but different manifestations of the same phenomena. In fact, electrical engineering books often write  $s = j\omega$  and treat these methodologies nearly interchangeably. We will do the same. Differentiating a time-domain function is said to be equivalent to multiplying with  $s$  in the transform domain, The (Laplace) transform domain is also called the frequency domain, since there is that  $\omega$ . Equivalently, integration in the time domain is equivalent to division by  $s$  in the frequency domain.

## 6.6 LET US CHECK

If we can indeed handle capacitors and inductors as easily as resistors, albeit having to work with complex numbers rather than real numbers, it would nonetheless simplify our efforts. Typical textbooks would give the mathematical rigor and then perhaps illustrate the ideas with a few examples. Here we do it a bit differently. We will construct circuits and first try to compute, and then actually measure. Again, it is best if you build these circuits and measure. The next best option is to simulate.

The two components are in series, so the total impedance would be  $Z_R + Z_C = R + \frac{1}{j\omega C}$ . If we measure the *AC voltage* at point A, it should be

$$V_A = V \frac{1}{R + \frac{1}{j\omega C}} \quad (6.14)$$



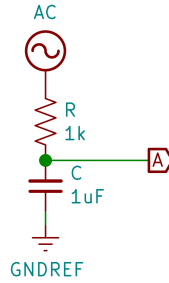


Figure 6.2: The Simplest RC Circuit.

where  $V$  is the supply voltage. If we use a multimeter set to measure AC voltages, then,

$$V_A = \left| \frac{1}{j\omega RC + 1} \right| V \quad (6.15)$$

Using  $\omega = 2\pi f$  we have,

$$V_A = \left| \frac{1}{j2\pi f RC + 1} \right| V \quad (6.16)$$

Let us set  $f = 100$ ,  $R = 1k$ , and  $C = 1\mu F$  as shown. Then,

$$V_A = \left| \frac{1}{1 + j0.2\pi} \right| V \quad (6.17)$$

The absolute value of  $1 + j0.2\pi$  is  $\sqrt{1^2 + 0.2\pi^2} = \sqrt{1.395} = 1.181$ . So,  $V_A = \frac{1}{1.181}V = 0.8467V$ . If  $V = 1V$ olt, then  $V_A = 0.8467V$ . We may repeat the computations for other frequencies.

Table 6.1:  $V_A$  as a Function of Frequency  $f$   
(see Figure 6.2)

$f$	$V_A$
50	0.954V
100	0.847V
150	0.728V
200	0.623V
250	0.537V

Well, so far so good. If the complex variable trick works, it would indeed simplify our life quite a bit. Here is another example to deepen our insights. Let us try adding one more node to our circuit.

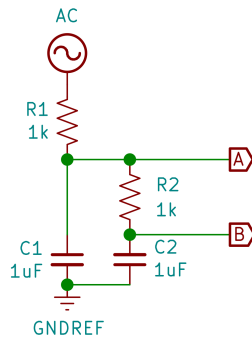


Figure 6.3: The Next Simplest RC Circuit.

Here we have two nodes, labeled A and B. Again, given that the supply voltage is 1V AC with a frequency of 100, let us compute the voltages at these nodes,  $V_A$  and  $V_B$ . The right portion of the circuit,  $R_2$  and  $C_2$  are in series. Their total impedance, given the component values and the frequency, is,

$$R_2 + \frac{1}{j\omega C} = 10^3 - \frac{j}{2\pi 10^2 \cdot 10^{-6}} = (10^3 - j \cdot 1.592 \cdot 10^3) \quad (6.18)$$

An experienced engineer, at this point, would probably switch impedance units from Ohms to kOhms. So, the impedance would be  $1 - j \cdot 1.592$  k.

This makes sense, as well as it is rather intuitive. We often deal with resistances in kOhm ranges. So, keeping the unit Volt the same, we may switch units to kOhms and milliAmperes. We also adjust the units of capacitance to this three-orders-of-magnitude change, and measure capacitance in milliFarads. The impedance using kOhms becomes  $(1 - j \cdot 1.592)k$ . Note that we designated the impedance by a simple "k", as the "Ohm" part is understood. This impedance is in parallel with that of  $C_1$ . Being parallel, we need compute the reciprocal of the sum of the reciprocal impedances. Let us first compute the reciprocal of  $(1 - j \cdot 1.592)$ .

The complex number in the denominator of a fraction is removed by multiplying both the numerator and the denominator by the complex conjugate of the denominator.

$$\begin{aligned} \frac{1}{1 - j \cdot 1.592} &= \frac{1 + j \cdot 1.592}{(1^2 + 1.592^2)} \\ &= \frac{1 + j \cdot 1.592}{3.533} && (6.19) \\ &= 0.2830 + j \cdot 0.4506 \end{aligned}$$

The reciprocal of the impedance of  $C_1$  is simply  $j \cdot 2\pi f \cdot C$  which is  $j \cdot 0.6283$ . Note that we are using nanoFarads as units here. Adding the two reciprocals we get  $(0.2830 + j \cdot 1.079)$ , whose reciprocal is once again computed using its complex conjugate. The computation yields  $(0.2274 - j \cdot 0.8671)$ . This is the combined impedance of  $C_1$ ,  $R_2$ , and  $C_2$ . Since this impedance is in series with  $R_1$ , we simply add the impedance of  $R_1$  to find the total impedance as  $(1.227 - j \cdot 0.8671)k$ .

Before we go any further, let us look at our result  $(1.227 - j \cdot 0.8671)k$  and see if it makes sense. The impedance, albeit a complex number, is above 1k. That is, the magnitude is 1.227k from the real (resistive) part plus  $-0.8671k$  from the complex component. This makes sense. The complex component comes from the capacitors. We may compute the magnitude of the complex number by squaring the two parts, adding them, and taking the square-root. This gives us 1.503k. The argument in radians is computed as the  $\arctan\left(\frac{-0.8671}{1.227}\right)$  or  $-0.6152$  rad.

*I always seek some order-of-magnitude equivalence between the resistors and capacitors, once the frequency is known. For example, if the frequency is 100Hz, then a 1μF capacitor would have an impedance of  $\frac{1}{2\pi \cdot 10^{-4}}$  Ohms, which equivalent a 1.6k resistor, and so on.*

The complex component, as well as the angle, is negative, indicating a lag of the signal – again, consistent with the nature of capacitance.

Given the circuit (Figure 6.3), with the impedance of the total load calculated as  $(1.227 - j0.8671)k$ . One may think of the voltage at node A as that produced by the voltage divider composed by  $R_1$  and the subcircuit composed by  $C_1$ ,  $R_2$ , and  $C_2$ . The voltage  $V_A$  would be,  $\frac{(0.2275 - j0.8673)}{(1.2275 - j0.8673)}$  times the input voltage. Dividing the two impedances is much easier in polar form. Converting the complex numbers to polar form, we express this as the ratio of  $0.8966/\underline{-1.314}$  to  $1.503/\underline{-0.6151}$ . Now it is matter of dividing the magnitudes to find the magnitude of  $V_A$ , that is,  $\frac{0.8966}{1.503} = 0.5965$ . The phase is computed by subtracting the denominator phase from the numerator phase. Specifically,  $-1.314 - (-0.6151) = -0.6963$ . If so desired, the result may be converted to rectangular form,  $0.5965/\underline{-0.6963} = 0.4576 - j0.3826$ . But, it is usually better to work with the magnitude of voltages, and thus, better to leave the results in polar form. As seen, addition and subtraction of complex numbers, as needed when impedances are in series, is easier in rectangular form, while multiplication and division is easier in polar form.

*It is a good idea to use a calculator that supports complex numbers. There are several good ones that work on phones.*

Next, we are ready to compute the voltage  $V_B$  at node B. We may consider  $V_B$  to be the voltage produced by the voltage divider  $R_2$  and  $C_2$ , excited by the now-computed voltage  $V_A$ .

$$\begin{aligned} V_B &= V_A \frac{Z_C}{R + Z_C} \\ &= V_A \frac{1}{R + \frac{1}{j\omega C}} \end{aligned} \quad (6.20)$$

We already crunched through much of the expressions here.

$$\begin{aligned}
 V_B &= V_A \frac{-j \cdot 1.592}{1 - j \cdot 1.592} \\
 &= V_A \frac{1.592 \angle -1.571}{1.880 \angle -1.010} \\
 &= V_A 0.7171 \angle -0.4504 \tag{6.21} \\
 &= (0.5965 \angle -0.6963)(0.7171 \angle 0.4504) \\
 &= 0.5051 \angle -1.2572 \text{ or } (0.1558 - j \cdot 0.4805)
 \end{aligned}$$

Computing and verifying voltages and currents gives us the confidence that we can apply the standard techniques to capacitors and resistors as long as we use complex numbers so that we keep track of not just the magnitudes, but also the phases. A few verification examples do not prove this statement, however, our intent here is to gain insights, not to pursue complete rigorous formal proofs. Incidentally, we did claim that phasor algebra works with inductors the same way, by using their impedance  $Z = j\omega L$ , where  $L$  is the inductance in Henrys. Let us next try an example with an inductor.

*We write Henrys and not Henries since the unit is a Henry, named after Joseph Henry (1797-1878).*

## 6.7 A TANK CIRCUIT

Figure 6.4 is affectionately known among old-timer radio enthusiasts as a *tank circuit*, with roots reaching all the way back to Leyden Jars, Electroscopes, Felix Savary, and Benjamin Franklin. It is called a tank circuit because it reminded people of how water sloshes back and forth when pushed periodically at the right tempo. The resistor in series is not critical. As such, electronics engineers would often refer to just the capacitor-inductor subcircuit as the tank.

If you compute the impedance  $Z$  of the parallel capacitor-inductor pair, you get a little bit of a surprise.

*I remember making an electroscope from some tin foil, thick copper wire, a big cork, and a small mayo jar back in elementary school. I was so proud of myself.*

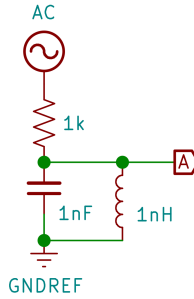


Figure 6.4: A Tank Circuit.

$$\begin{aligned}
 \frac{1}{Z} &= \frac{1}{Z_L} + \frac{1}{Z_C} \\
 &= \frac{1}{j\omega L} + j\omega C \\
 &= \frac{-j}{\omega L} + j\omega C \\
 &= \frac{j(\omega^2 LC - 1)}{\omega L} \\
 Z &= \frac{\omega L}{j(\omega^2 LC - 1)} \\
 &= \frac{j\omega L}{1 - (\omega^2 LC)}
 \end{aligned} \tag{6.22}$$

Incidentally, the reciprocal of the summed reciprocals is related to the *harmonic mean* of (values of) the parallel branches. If we were to divide the summed reciprocals by the number of branches, or equivalently multiply the final result by the number of branches, we would have the harmonic mean. Anyway, there you have it, for a given inductor and a capacitor, there is a certain frequency at which the impedance becomes infinitely large. Essentially, at that frequency, the capacitor-inductor pair acts like an open circuit. Since infinity is a very large amount of impedance, you would expect that the frequency-impedance graph would have an asymptotically raising peak at that particular frequency. The frequency, known as the *resonance frequency* is easily computed as, when the denominator reaches zero.

$$1 - (\omega^2 LC) = 0$$

$$\omega^2 LC = 1$$

$$\omega^2 = \frac{1}{LC}$$

$$\omega = \sqrt{\frac{1}{LC}} \quad (6.23)$$

$$2\pi f = \frac{1}{\sqrt{LC}}$$

$$f = \frac{1}{2\pi\sqrt{LC}}$$

The quantity  $\sqrt{LC}$  always appears to me as the *geometric mean* of (the values of) L and C. For the given circuit 6.4, the resonance frequency is the reciprocal of  $2\pi 10^{-9}$  which is 159.155 MHz ( $\approx 160$  MHz).

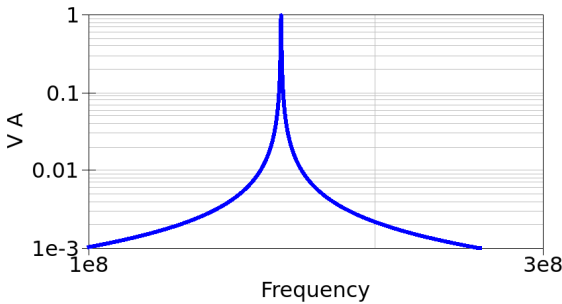


Figure 6.5: The *Tank Circuit*,  $V_A$  as a Function of Frequency.

Figure 6.5 is a log-log graph. You can see that the voltage  $V_A$  is 1 at the resonance frequency. At around 160 MHz, the tank circuit acts like an open circuit. Our insight that the voltages must asymptotically raise around the resonance frequency is also observed. As mentioned, the series resistor does not add much to the circuit, especially around the resonance frequency. At that frequency, the circuit appears to be broken, and thus, no matter the value of R, the voltage  $V_A$  would be the same as the supply voltage.

Often I get the feeling that electronics is symmetrical. There are many dualities. There is, after all, similarities between the impedances

of capacitors and inductors. When we place inductors in series, they have the same *feel* as capacitors placed in parallel. What would your instincts say if we built the tank circuit in series rather than in parallel? That would be a good exercise to pursue.

## 6.8 LET US BUILD AN OSCILLATOR

I got an idea<sup>2</sup>. After all, ultimately, I would say that engineering is about design. Referring back to Figure 6.3, I see that the signal at node B has a phase lag that is a function of the component values. Let me first try something and see if it works. Then I will explain what inspired me.

Figure 6.6 seems innocuous enough, but generates a periodic output from no discernible input. It is an oscillator.

*“Engineering is the art of directing the great sources of power in nature for the use and convenience of man.”* Thomas Tredgold (1788-1829)  
English engineer.

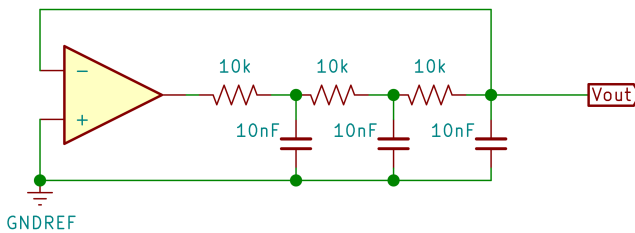


Figure 6.6: A Simple Oscillator.

Moreover, the opamp seems to be configured in a negative feedback mode, so you might think that the output would behave nicely and not oscillate at all. But it does, as shown in Figure 6.7.

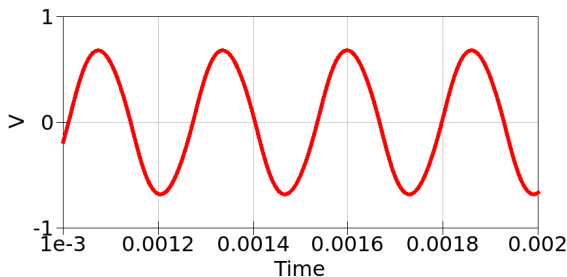


Figure 6.7: The Output Voltage of the Simple Oscillator.

<sup>2</sup> Actually, I first had this idea as a teenage hobbyist in high school in 1971. I was so proud of my discovery.



Can you figure out what is happening? It has not much to do with voltages, but rather with the phase. The feedback circuit, inspired by [Figure 6.3](#) introduces a phase difference. We computed the phase lag for given values in [Section 6.6](#). Here I used slightly different values, but still, there is a phase lag which we can compute or measure, or even simulate and observe. At a certain frequency, the phase difference becomes  $\pi$  radians. That is, the same signal but with an opposite sign. So, while the opamp looks like it is operating in a negative feedback configuration at most frequencies, there is a special frequency for which the phase difference introduced by the feedback network becomes  $\pi$  ( $180^\circ$ ). At that frequency, the feedback is positive not negative.

If you simulate this circuit with ideal components, you must start one of the capacitors with a little charge. Otherwise the circuit would be happy keeping all nodes at zero volts. In real life, there is no need to place a start-up charge, since the imperfections of the components (such as the opamp offset error) will start the circuit. Who says imperfection is bad? Since the circuit starts due to an ever-so-slight deviation from ideal equilibrium, it needs a little time to reach steady state. Here, the graph starts not at time zero, but after 10 milliseconds. *Reminds me of wabi-sabi.*

How would you compute the frequency generated by the oscillator? It is the frequency at which the phase of the impedance of the feedback circuit is  $\pi$ . You may write the phase as a function of the component values and the unknown frequency, and solve for the frequency that would make the phase exactly  $\pi$ . We will not do that. I assume by now we have some insights into circuits and some engineering intuition. If you ever wondered how experienced engineers quickly find a way to read the circuits, what I say now may give you some idea. Look at the circuit ([6.6](#)) and note that there are three sets of resistor-capacitor Subcircuits. Each one could be thought of as a voltage divider ([Figure 6.8](#)).

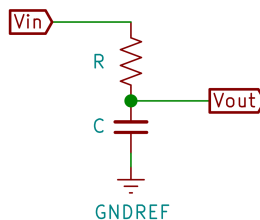


Figure 6.8: The Voltage Divider Subcircuit.

We are not as much interested in the magnitude of the output voltage as we are in the phase lag it introduces. The output voltage  $V_{out}$  as

a function of the input voltage  $V_{in}$  is easily computed. After all, this is nothing but a voltage divider. The impedance of the capacitor is  $\frac{1}{j\omega C}$  and the total impedance is  $R + \frac{1}{j\omega C}$ . So,

$$\begin{aligned}
 V_{out} &= V_{in} \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \\
 &= V_{in} \frac{-j}{\omega RC - j} \\
 &= V_{in} \frac{1 - j\omega RC}{1 + \omega^2 R^2 C^2}
 \end{aligned} \tag{6.24}$$

We said we were not much interested in the magnitude of the output voltage, so let us ignore the real-valued denominator. The phase will be the argument of  $1 - j\omega RC$ . Unless the frequency, or  $R$ , or  $C$  is zero, there will be a phase lag in the range  $(0, -\frac{\pi}{2})$ . Note that the interval boundaries are excluded from the set of possible values.

Now here comes the key insight: if you cascade these voltage dividers, each will introduce a phase lag which are additive. In our circuit (6.6) we have three identical voltage dividers. As we want the total phase lag to be  $\pi$ , which of course is the same as  $-\pi$ . If each voltage divider had a phase lag of  $\frac{\pi}{3}$  the total lag would be exactly what we need. Of course, we do not set the lag. The circuit settles at at the frequency at which the lag at each divider is  $\frac{\pi}{3}$ .

$$\begin{aligned}
 -\frac{\pi}{3} &= \arg(1 - j\omega RC) \\
 \tan\left(-\frac{\pi}{3}\right) &= -\omega RC \\
 \omega RC &= \sqrt{3} \\
 2\pi f RC &= \sqrt{3} \\
 f &= \frac{\sqrt{3}}{2\pi RC}
 \end{aligned}
 \tag{6.25}$$

Let us find out the lag of each divider<sup>3</sup> in our circuit. With  $R = 10\text{k}$  and  $C = 10\text{ nF}$  the frequency comes to about 2.7 kHz. Well well, there is good news and there is bad news. Good that we got a nice sinusoidal signal out with no other input. It looks clean. We can sure use it. But the bad news is that, looking at [Figure 6.7](#), the frequency is more like 3.7 kHz. It seems our assumption that these three voltage dividers are independent is a bit of an optimistic overstatement. After all, the downstream dividers are in parallel with the capacitor of each divider. An exact expression is possible, but we took a shortcut, did we not? Still, not all is lost. After all, electronics insights is both qualitative and quantitative – perhaps mostly qualitative. Our hunch that three voltage dividers, each with a lag in the range of zero to  $\frac{\pi}{2}$ , when cascaded could create a combined lag of exactly  $\pi$ , and thus fool the opamp as it seems like positive feedback. Not only did we have the insight, but we as good engineers, found a practical use for this – namely, we built a nice oscillator. If you want to get closer to the computed frequency, I suppose we need to not load the stages, and make them really independent. This is another good opportunity to revisit some previous discussions. Do you remember the good old voltage follower?

Buffering the voltages dividers is a job for the voltage follower. As seen in [Figure 6.9](#), the extreme high input impedance of the opamp means that there is no *leakage of current* from the dividers to the following stages. The voltage at each divider is faithfully duplicated and presented to the downstream stage all with low impedance (read: power to drive the following stage).

<sup>3</sup> Notice we just said “divider” and not the whole term, “voltage divider”. Field engineers more often than not drop part of the terminology when it is obvious from the context. You might as well get used to it too.

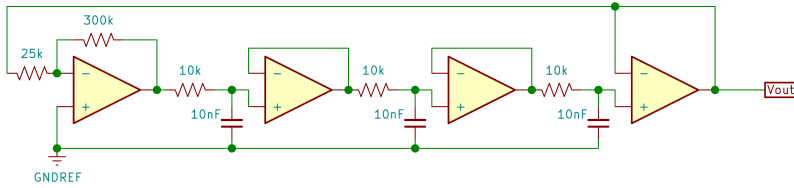


Figure 6.9: A Simple Oscillator.

Now the frequency seems closer to the 2.7 kHz we computed. You may wonder why we have an opamp at the end. Well, if we had connected the output directly to the initial amplifier, then the 25k input resistor at the amplifier would drain (leak current from) the capacitor at the last stage. After all, this resistor is connected to the inverting input of the initial opamp, whose non-inverting input is connected to ground. Since there is no voltage difference between the two inputs of the opamp, the 25k resistor is as good as having its end connected to ground. A 25k resistor in parallel with the capacitor of the last stage would alter its characteristics, so we put another voltage follower there. After all, many modern opamps come 4 in a single chip. The good-old LM747 was two LM741s in a single 14-pin DIP (Dual in-line Package), and the LM148, a quad version of the LM741. I used the workhorse LM324, the improved version of the quad 741 in 14-pin DIP in many a circuits I designed. By the way, it seems the LM747 came out just about the same time Boeing made a big splash with its Jumbo 747.

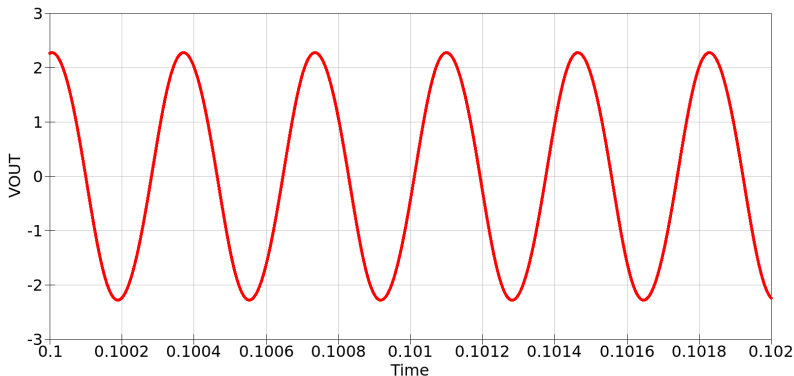


Figure 6.10: The Output Voltage of the Simple Oscillator. Voltage followers are used as in [Figure 6.9](#).

## 6.9 EXPERIMENTATION AND REFLECTION

1. Build the 6.3 or at least simulate it. Verify that our computations match the measured value of  $V_A$ . Compute the voltage  $V_B$  and verify its value on the circuit as well.
2. Build the tank circuit in series as shown in Figure 6.11. Write the equations that describe the voltage  $V_A$  at node A as a function of frequency. Comment if you see any similarities to the parallel tank circuit.

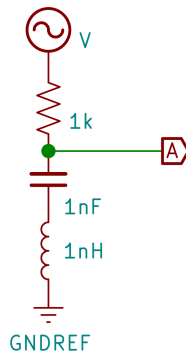


Figure 6.11: A Series Tank Circuit.

3. Experiment with the oscillator (Figure 6.6) by changing the gain at the first opamp. See how the circuit behaves if the gain is not sufficient to sustain oscillations.
4. Since many opamps now come in four in a package, we may build an oscillator with four-stages, each stage introducing a lag of  $\frac{\pi}{4}$  radians. This type of oscillator has become known as the *bubba oscillator*. Write the equations following the same train of thought in this chapter. Then build and compare your observations to your calculations.
5. There are several well-known topologies used to make opamp-based oscillators. Search the web to find out about Colpitts Hartley, quadrature, and Wien-bridge oscillators. Build and experiment with these oscillators. Discuss the advantages and disadvantages of each.

*It is embedded in electronics folklore that some mischievously playful applications engineers at Texas Instruments coined the word bubba.*

6. Revisit the Sallen-Key Topology (Figure 6.12). Write the output as a function of the input using general components. That is, allow the four components to have general impedances, allowing each to be a resistor, a capacitor, an inductor. How do various combinations of the component type selections affect the behavior of the circuit?

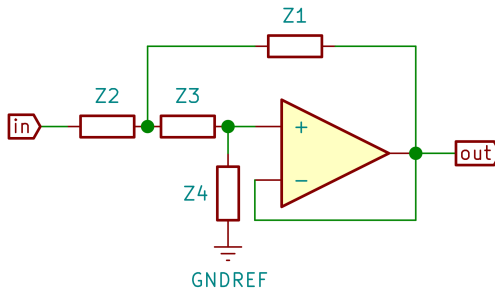


Figure 6.12: The Sallen-Key Topology.

TRANSFER FUNCTIONS

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If you have come so far, either you are interested or have nothing better to do. Or perhaps both.

Formalisms are good because they lead to standard methodologies which, when adopted and become familiar, allows the engineer to cruise on auto-pilot. We know how to evaluate circuits using complex-algebra-based phasors which makes life easy. You come to a point where you do not even think about what is complex, what is an angle, nothing. You follow the methodology and use it. It lets you focus on the real issues – in my case, designing circuits. Sure, you could use canned software packages, but going through the motions, preferably at a lab workbench, provides invaluable insights. I suppose it is also a matter of how you view electronics. Is it just a job or a passion, is it rather mundane pre-scripted work you are after, or imaginative design work? Canned packages are probably fine if electronics is not the main ambition in your life. After all, one may gain insights into the software packages, learn its quirks and idiosyncrasies, find shortcuts, etc. But if you have come this far, let us assume that you are still anxious to continue with our journey, gain further insights, and see some more of the landscape.

Electrical engineering has almost a two-century footprint in academia. Over the years besides our trusted phasor formalism, there has been wide-spread acceptance of another key concept: transfer functions using Laplace transforms. You see, when we limit ourselves to sinusoidal signals, there is much structure that is open to intellectual and computational exploitation. For one thing, as said, all signals in well-behaved circuits with linear components<sup>1</sup> will have the same frequency. If you have a keen mathematical eye, you will of course notice that both the derivative and the integral of a sinusoidal function are also sinusoidal. So, just as in phasors, if we want to deal with differentiation or integration, once again, the results may be obtained not by explicitly applying calculus to the equations, but by shortcuts we develop. It is not surprising that the Laplace transform simplifies things. Things *are* simpler than the generic case, due to such structure. Another way

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<sup>1</sup> see [Section 2.3](#)

to see this is to realize that we are dealing with a subset of all possible time-varying functions. And in mathematical jargon, the set of functions under differentiation and integration is closed. They are all sinusoidal.

## 7.1 TRANSFER FUNCTIONS

Modern control theory views circuits as boxes with input signals and output signals. A filter is a good example. The output is a function of both the input and what happens in the box. It turns out that if we describe this relationship in the transform domain using the Laplace transform, things become easier. Essentially, differential equations are transformed into algebraic equations. Typically, the relationship is depicted as below.

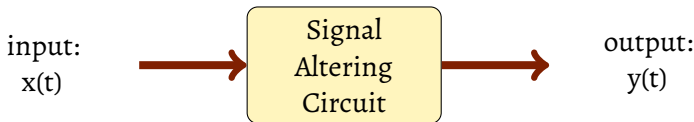


Figure 7.1: The Circuit as a Black Box.

The input and output signals are shown as  $x(t)$  and  $y(t)$  in [Figure 7.1](#). These are typically the input and output voltages, and considered mostly as either DC or sinusoidal signals. Each *nice* continuous function has a transform<sup>2</sup>, typically denoted by the same letter as the time-domain function, but with its name capitalized. You will recall from our discussions in [Section 6.5](#) that differentiation amounted to multiplying with  $s = j\omega$ . Here, the  $s$  is the variable of the transform domain version of the signal. But there is more. The ease of differentiation allows us to write differential equations in time domain as simple algebraic equations in the transform domain. Solving differential equations becomes easier – but that is not the point. It also allows us to *think* in the transform domain. We manipulate simple linear algebraic equations but are conscious of their meaning as differential equations. Since the variable of the transforms  $s$  is related to frequency, electrical engineers refer to the transform-domain as *frequency-domain*. It seems it is more intuitive that way. I realize, I appeal to your intuition rather than a formal development here. The subject of transforms is quite elegant and has many many applications in engineering. The purpose here is to develop intuition and gain some operational experience. This,

2 Notice again, we dropped the term “*Laplace*” and just said “transform”.



I hope, will constitute the motivation for you, in the fullness of time, to deep-dive into transforms.

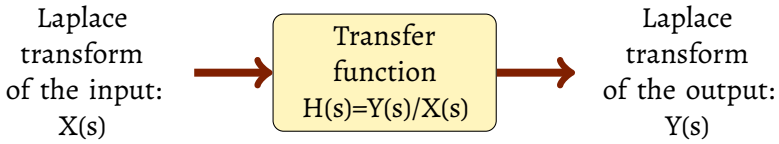


Figure 7.2: The Circuit as a Black Box in the Transform Domain.

We revisit our black box but now consider it in the frequency domain. Since the input and output are now described algebraically, their ratio  $H(s)$  is easily evaluated. This is called the transfer function. This  $H(s)$  uniquely describes what the circuit does, how precisely it takes the input to produce the output. A good example is our filters, which take input sinusoidal signals and apply selective gain or attenuation based on the frequency of the signal to produce the output.

## 7.2 TIME FOR SOME EXAMPLES

Let us state the trivial one first: What is the transfer function of an inverting amplifier? Assuming it is an ideal opamp, as in [Figure 2.1](#), it is simply  $H(s) = -\frac{R_2}{R_1}$ , is it not? You want the output, multiply the input by the constant  $-\frac{R_2}{R_1}$ . Luckily, the transform domain preserved linearity and superposition (see [Section 2.3](#)).

That was easy. Now let us take a look at the figure we discussed ([Figure 4.6](#) in [Section 4.5](#) and ask, *what is its transfer function?* Here is the circuit once again, this time with component values ([Figure 7.3](#)).

Let us compute the impedance  $Z_f$  of the feedback subcircuit composed of  $R_2$  and  $C$  in parallel.

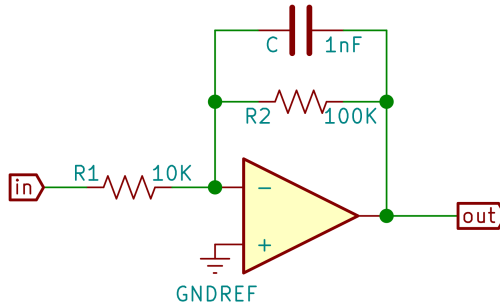


Figure 7.3: Low-Pass Filter.

$$\begin{aligned} \frac{1}{Z_f} &= \frac{1}{R_2} + sC \\ &= \frac{1 + sR_2C}{R_2} \end{aligned} \quad (7.1)$$

$$Z_f = \frac{R_2}{1 + sR_2C}$$

So, the ratio of the output to the input is,

$$\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = -\frac{R_2}{R_1 + sR_1R_2C} \quad (7.2)$$

$$H(s) = -\frac{R_2}{R_1 + sR_1R_2C}$$

With the values inserted, we have

$$\begin{aligned} H(s) &= -\frac{100\text{k}}{10\text{k} + s \cdot 10\text{k} \cdot 100\text{k} \cdot 1\text{n}} \\ &= -\frac{10^5}{10^4 + s10^4 \cdot 10^5 \cdot 10^{-9}} \\ &= -\frac{10}{1 + s10^{-4}} \end{aligned} \quad (7.3)$$

At this point, you may get this eerie feeling that something is amiss. Take a look at the transfer function. The ratio must be unitless. It is, after all, the ratio of the output to the input, both the same units, whatever they may be. The right hand side, has Ohms in the numerator, and the sum of Ohms ( $R_1$ ) with whatever units  $sR_1R_2C$  has. As we discussed in [Section 2.4](#),  $R_2C$  is a time constant and thus has units of time ( $[T]$ ). If the dimensional integrity is to be preserved,  $s$  must have dimensions  $[T]^{-1}$ . Could that be it? Again, recall how we defined  $s = j\omega$  when we introduced the phasor notation,  $V(t) = Ae^{j\omega t}$ . Now, a good engineer would understand that if there is an exponent, it better be unitless. This is clear if you consider the Taylor series expansion of the exponential and realize that all powers of the exponent contribute to the exponentiation. Speaking of symmetry, the inverse, that is if there is a logarithm of an expression, it too better be unitless. So, yes,  $s$  has units  $[T]^{-1}$ . It, after all, is a multiple of the frequency. You see, we switched our viewpoint from time ( $t$ ) to (a multiple of) frequency ( $s$ ), from the time domain to the transform domain. When you realize the link, you better appreciate how practical former electrical engineers have been, all for the sake of facilitating comprehension and computation. Maybe it was because computation was time consuming before the abundant availability of computers.

*Actually,  $s = -j\omega$ , but the sign does not really matter. It makes the phasor rotate in the opposite direction, that is all.*

First, let us look at the transfer function ([Equation 7.3](#)) from a pragmatic viewpoint. At a glance, it gives us the information that the circuit has zero gain at high enough frequencies, and a gain of  $-10$  at low frequencies. Why? Well,  $s = j\omega = j2\pi f$ , is it not? Just let  $s \rightarrow 0$  and then let  $s \rightarrow \infty$  and ask what is the value of  $H(s)$  – it is a constant at these extremes. The negative comes from the fact that the opamp is run in an inverting configuration. The time constant is  $10^{-4}$  so we are looking at the millisecond range of operation, or equivalently, interesting things, such as the cut-off frequency, in the kHz range.

More rigorously, though, We see that the magnitude of the transfer function [7.2](#) will always be in the range  $(0, \frac{R_2}{R_1})$ , depending on the frequency. Making the substitution  $s = j\omega$ , we express the transfer function as a complex number.

$$\begin{aligned}
 H(s) &= -\frac{R_2}{R_1 + j\omega R_1 R_2 C} \\
 &= -\frac{R_1 R_2 - j\omega R_1 R_2^2 C}{R_1^2 + \omega^2 R_1^2 R_2^2 C^2}
 \end{aligned}
 \tag{7.4}$$

The negative imaginary component says that the output will lag the input. The angle is determined by the numerator alone, since the denominator is a real value.

$$\begin{aligned}
 \phi &= -\arctan\left(\frac{\omega R_1 R_2^2 C}{R_1 R_2}\right) \\
 \phi &= -\arctan(\omega R_2 C)
 \end{aligned}
 \tag{7.5}$$

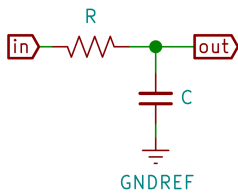
But we knew all this. We could have computed these as we discussed our filters. What exactly does the transfer function contribute? Well, if we could start with a transfer function and then build a circuit that implements it, we could save some time. Linear circuits will have transfer functions in the form of ratios. Both the numerator and the denominator will be polynomials in  $s$ . The zeros of the numerator polynomial will tell us when the circuit has zero impedance. These are, appropriately, called the *zeros of the transfer function*. Similarly, the zeros of the denominator polynomial tell us for what frequencies the circuit has infinite impedance. These are called the *poles of the transfer function*. Let us say that we want a filter that has infinite impedance at a certain frequency. Sounds familiar, does it not. Think of the tank circuits we build (6.4). Notice it has one pole and no zeros.

### 7.3 A PRACTICAL VIEW OF TRANSFER FUNCTIONS

All this discussion is fine, but I must say, experienced practicing engineers often have a more pragmatic view of transfer functions. Let me explain.

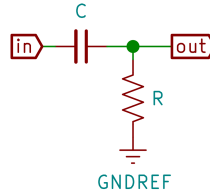
First, you notice that anytime there is a capacitor or an inductor, we add one more order to the differential equations. Capacitors and inductors are energy-storing passive linear elements. They are linear, because their behavior is expressed by rather simple linear differential

equations. This is the case at least if the circuit is properly built. Sure, you may place two capacitors in parallel and your differential equations will have the same order, not an order higher. After all, placing two capacitors in parallel simply makes a bigger capacitor with a capacity equal to the sum of its constituent capacities, does it not? So, the order is the same as the *effective* number of linear energy-storing passive components. Next, we see that the transfer function, when written in the frequency domain using the Laplace transform, typically ends up looking like a ratio of two polynomials in  $s = j\omega$ . The transform is useful, since it allows differential equations to be represented by simple algebraic equations – in most cases polynomials. The magnitudes of the numerator and the denominator depend on  $s$  or more revealingly, the frequency,  $\omega = 2\pi f$ . The frequencies at which the numerator or the denominator magnitude is zero are of interest. These are the zeros and poles of the system.



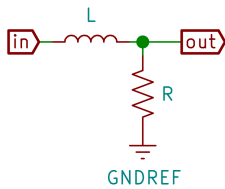
(a) Low Pass RC Voltage Divider

$$H(s) = \frac{1}{sRC + 1}$$



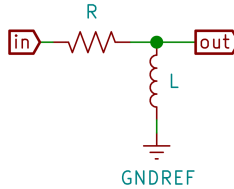
(b) High Pass RC Voltage Divider

$$H(s) = \frac{sRC}{sRC + 1}$$



(c) Low Pass RL Voltage Divider

$$H(s) = \frac{R}{sL + R}$$



(d) High Pass RL Voltage Divider

$$H(s) = \frac{sL}{sL + R}$$

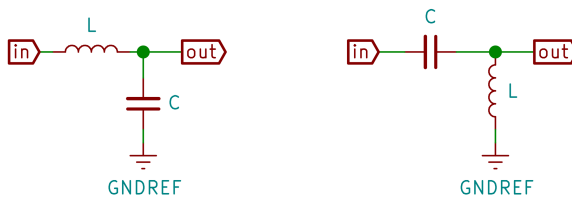
Figure 7.4: R-C Voltage Dividers.

Besides the insights into the form of the transfer function, we have some ideas about what kinds of transfer functions simple *building block* circuits have. Our trusted good-old voltage divider is once again a good place to start (Figure 7.4). We will call these circuits *elemental circuits*. They will be used as the smallest of all building blocks to construct many complicated circuits. When experienced engineers *read* circuits, they often view these voltage dividers not as individual components, but as atomic blocks.

*This is a bit like musicians often viewing chords as elemental units and not as individual notes.*

Each circuit has one energy-storing element, and thus the denominator polynomials are of degree 1. The numerators depend on the type of circuit. Noting that the variable  $s$  is proportional to frequency, think about what happens when  $s = 0$ , that is under DC voltages. The low pass circuits will have a gain of 1, and the high-pass circuits will have a gain of 0. What about when  $s \rightarrow \infty$ ? Then it is the other way around. All this insight without any effort at all.

We may extend these simple building blocks by using a capacitor and an inductor, as depicted in Figure 7.5. As we will have two energy-storing components, we would expect to see second-degree denominator polynomials. An experienced engineer could quickly determine, from the characteristics of the circuit, and the number of effective energy-storing elements, the degrees of the numerator and denominator polynomials. The degree of the numerator polynomial in passive circuits, like the voltage-divider-type we see here, will always be at most the degree of the denominator polynomial.



(a) Low Pass LC Voltage Divider

$$H(s) = \frac{1}{s^2LC + 1}$$

(b) High Pass RC Voltage Divider

$$H(s) = \frac{s^2LC}{s^2LC + 1}$$

Figure 7.5: L-C Voltage Dividers.

Before we end this section, let us not forget our tank circuits. Circuit 7.6a passes most low frequencies through the inductor and high frequencies through the capacitor. Only at the resonance frequency of

the tank circuit, does the circuit deny any current. The opposite is the case for Circuit 7.6b. It shorts most frequencies to ground, except at the resonance frequency, which is seen at the output.

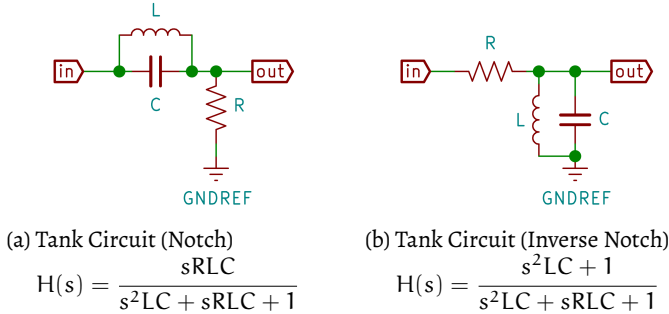


Figure 7.6: Tank Circuit Voltage Dividers.

7.4 TRANSFER FUNCTIONS ARE LINEAR OPERATORS

Take a look at the elemental circuits in Figure 7.4, Figure 7.5, and Figure 7.6. Each has its transfer function. If we cascade two of these building blocks, the combined transfer function would be the product of the two individual transfer functions. After all, the input times the transfer function gives the output.

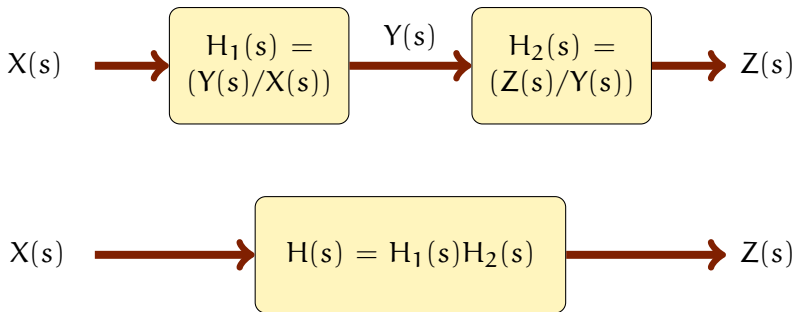


Figure 7.7: Cascading Transfer Functions.

As seen, cascading transfer functions simply amounts to multiplying the constituent transfer functions, that is,  $(Y(s)/X(s))(Z(s)/Y(s)) = (Z(s)/X(s))$ . Along with superposition (see Section 2.3), that is, the additive property, we see that transfer functions make short work of circuit analysis and design.

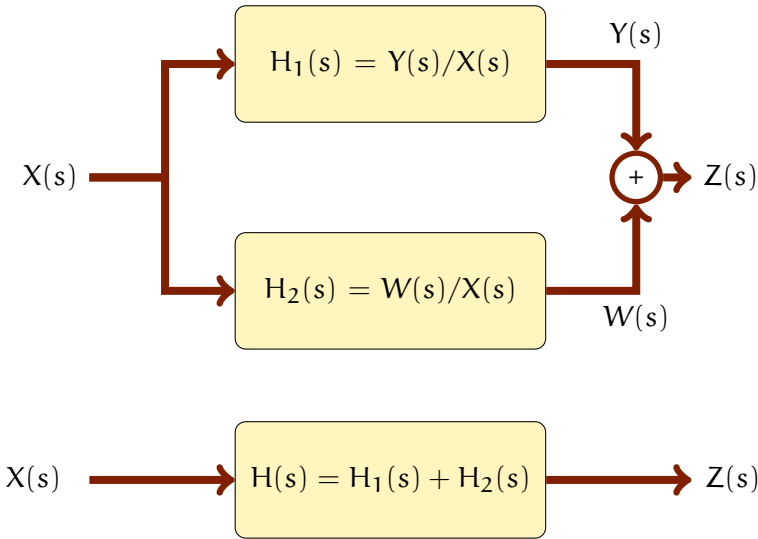


Figure 7.8: Adding Transfer Functions.

Even though we showed only two transfer functions cascaded, it should be clear that this is extensible. The same goes for adding or subtracting transfer functions. In fact, any linear combination of these operations is permissible. With this in hand, we can design circuits by combining elemental ones, the building blocks based on the voltage dividers that have served us so well so far.

## 7.5 SYNTHESIS – CIRCUIT DESIGN

The title sounds quite pretentious, we do not claim to be able to design *any* circuit, but simple circuits intended to run with sinusoidal signals. I am sure you can, at this point, take a circuit, and write its transfer function. So it is time to entertain the opposite. Suppose you are given the transfer function and asked to design a circuit that implements the functionality specified by it.

We have elemental circuits, which we may string to our heart's content and keep multiplying the transfer functions until you see the proper degree polynomials at the denominator and the numerator. We may also add these to get even more flavors. Since there are various ways we can obtain the same transfer function, we realize that there is more than one way to provide a circuit for a desired transfer function. Here is a nice one that would serve as an example. Let all coefficients



of the numerator ( $N_i$ 's) and the denominator  $D_i$ 's be real numbers within reason.

$$H(s) = \frac{sN_1 + N_0}{s^2 + sD_1 + D_0} \quad (7.6)$$

Note that we set the coefficient of the highest power of the denominator polynomial equal to unity. This is good engineering practice. Engineering parsimony says that use only as many parameters as needed. If the leading coefficient were not unity, we would normalize by dividing both the numerator and the denominator by this value and end up with the form of Equation 7.6. *If you want to impress your friends, refer to this as a monic polynomial.*

Granted, there may be multiple ways to design this circuit, but let us give it a try. First, observe that this is a low-pass-type of circuit. How do we know that? When  $s \rightarrow \infty$  the gain is 0 and when  $s \rightarrow 0$ , the gain is  $\frac{N_0}{D_0}$ . Good. Now also observe that the denominator polynomial is of degree 2. We should expect that two energy-storing components would suffice to build the circuit. Engineering insight helps us, does it not?

A common trick is to write the expression 7.6 as a sum of two fractions.

$$\begin{aligned} H(s) &= \frac{sN_1 + N_0}{s^2 + sD_1 + D_0} \\ &= \frac{u}{s - r_1} + \frac{v}{s - r_2} \end{aligned} \quad (7.7)$$

where  $r_1$  and  $r_1$  are readily evaluated as the roots of the original denominator polynomial  $s^2 + sD_1 + D_0$ . This well-known technique is called *partial fractions expansion*.

Back to our design. Pick two elements from Figure 7.4 such that when we add their transfer functions, we get a polynomial of  $s$  of degree 1 in the numerator, and of degree 2 in the denominator. As seen, and as predicted, there are alternatives. Let us pick circuits a and c from Figure 7.4. If we want a clean circuit, we must make sure that subcircuits do not load each other. Otherwise, we will have the same error we saw when making our oscillator in Section 6.8. You may first try this and see how bad it will be. Then try to compensate by slightly adjusting the component values. But let us use opamps –

after all our whole subject is opamps – and make sure that subcircuits operate as intended, without affecting each other. There is actually one other advantage to using opamps. The two fractions we add may also need different gains. Let us clarify.

If we add the transfer functions of these two elemental circuits, we get,

$$H_a(s) = \frac{\frac{1}{R_1 C}}{s + \frac{1}{R_1 C}} \quad (7.8)$$

$$H_c(s) = \frac{\frac{R_2}{L}}{s + \frac{R_2}{L}}$$

But we can also multiply each transfer function by a constant. This would correspond to introducing a gain of an arbitrary value, positive or negative. If the gain is to be greater than unity, the opamp is an ideal choice. If the gain is less than unity, then we may use a simple voltage divider.

$$H(s) = G_a H_a(s) + G_c H_c(s)$$

$$= G_a \frac{\frac{1}{R_1 C}}{s + \frac{1}{R_1 C}} + G_c \frac{\frac{R_2}{L}}{s + \frac{R_2}{L}} \quad (7.9)$$

$$= \frac{s\left(\frac{G_a}{R_1 C} + \frac{G_c R_2}{L}\right) + (G_a + G_c) \frac{R_2}{R_1 C L}}{s^2 + s\left(\frac{1}{R_1 C} + \frac{R_2}{L}\right) + \frac{R_2}{R_1 L C}}$$

We are almost there. Now we need to pick component values that would give us the specified transfer function. That is, we pick the values in [Equation 7.9](#) that would coincide with the coefficients in [Equation 7.6](#);

$$\begin{aligned}
 N_1 &= \frac{G_a L + G_c R_1 R_2 C}{R_1 C L} \\
 N_0 &= (G_a + G_c) \frac{R_2}{R_1 C L} \\
 D_1 &= \frac{L + R_1 R_2 C}{R_1 C L} \\
 D_0 &= \frac{R_2}{R_1 C L}
 \end{aligned} \tag{7.10}$$

Since we have six parameters which we may freely choose, but only four coefficients, we may select the same value for  $R_1$  and  $R_2$ . This still gives us one more degree of freedom than needed. Since our equations are not linear, strictly speaking, this may not work in general, but here, the equations look innocent enough to give it a try.

$$N_1 = \frac{G_a L + G_c R^2 C}{R C L} \tag{7.11}$$

$$N_0 = (G_a + G_c) \frac{1}{C L} \tag{7.12}$$

$$D_1 = \frac{L + R^2 C}{R C L} \tag{7.13}$$

$$D_0 = \frac{1}{C L} \tag{7.14}$$

Let us try a numerical example. We expect  $R$  to be in the kiloOhm range,  $C$  in microFarads and  $L$  in milliHenrys. Keeping these in mind, here is a reasonable transfer function with round numbers.

$$\begin{aligned}
 H(s) &= \frac{s N_1 + N_0}{s^2 + s D_1 + D_0} \\
 &= \frac{s(2.1 \cdot 10^4) + 2.2 \cdot 10^7}{s^2 + s(2.0 \cdot 10^4) + 2.0 \cdot 10^6}
 \end{aligned} \tag{7.15}$$

Since we have one degree of freedom, let us pick the capacitor to be  $C = 10\mu\text{F}$ . The value of the inductor  $L$  may now be computed from Equation 7.14 as  $L = 50\text{mH}$ . The DC gain,  $\frac{N_0}{D_0} = 11$ , which is the same as  $G_a + G_c = 11$ . We now have a quadratic formula in Equation 7.13 to compute  $R$ . All values other than  $R$  are known, as we compute  $R = 995\text{ Ohm} \approx 1.0\text{k}$ . The only other thing left is to compute  $G_a$  and  $G_c$ . Their sum was computed from the DC gain as 11, we use Equation 7.11 and the DC gain to compute the gains. Note however, that  $G_a L$  is many orders of magnitude smaller than  $G_c R^2 C$ , which means,

$$\begin{aligned} N_1 &\approx \frac{G_c R^2 C}{RCL} \\ &\approx \frac{G_c R}{L} \end{aligned} \tag{7.16}$$

which gives  $G_c \approx 1$ , and thus,  $G_a \approx 10$ . Having found the component values, we now may build and test our circuit (Figure 7.9).

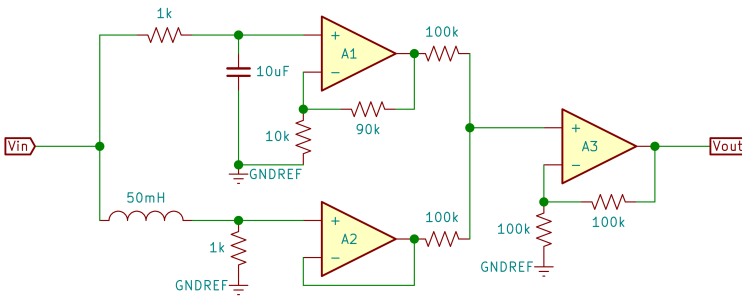


Figure 7.9: A Designed-to-Order Circuit.

The two elemental circuits are fed from the same AC source. The top circuit is circuit a from Figure 7.4, and the bottom, circuit c. The output of the top circuit is amplified by a factor of 10. The output of the bottom circuit feeds into a voltage follower, so the gain there is unity. Both opamp outputs are added by a third output opamp.

The frequency response as shown in Figure 7.10 indeed looks like the circuit works as a low pass filter. The second-order nature of the filter is evident from the two cut-off frequencies. The red line shows

the output, the blue and green lines show the components from the top and bottom half of the contributing circuits. The gains of 10 and 1 are clearly visible on the graph.

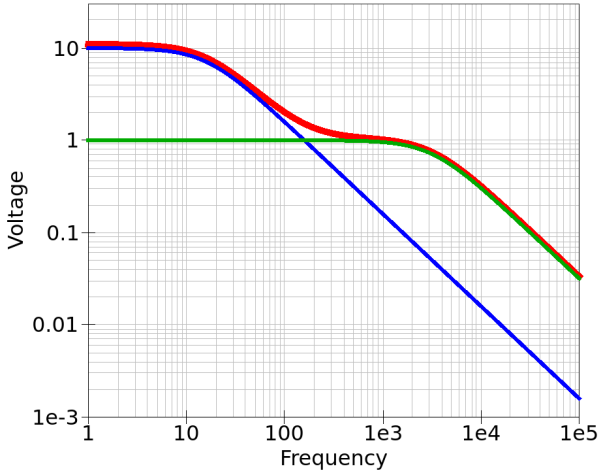


Figure 7.10: Frequency Response of the Designed Circuit.  
(blue and green lines are from the subcircuits a and c,  
the red line is the combined frequency response.)

As stated, there are many other alternatives you could choose. Pragmatic concerns must be kept in mind. Keep the capacitor and inductor values reasonable. If other factors, such as noise, vibration, cost, etc. are of concern, then of course, there is much more to consider. Nonetheless, the exercise should give you an idea. Moreover, you may consider not just adding transfer functions, that is, not just placing elemental circuits in parallel, but other network topologies. There is no limit to how you may arrange the elemental circuits.

## 7.6 THE LAPLACE TRANSFORM

Since we made use of the Laplace transform, it may be a good idea to say a few things about this rather useful tool. It is named after Pierre-Simon, marquis de Laplace (1749-1827), the French polymath, but its practical use in engineering and other applied areas owes to the development and contributions of many scholars.

A transform deals with two domains which have a one-to-one relationship. Elements and operations in one domain have corresponding

elements and operations in the other. Some operations are easier in the other domain, so you transform your elements to the transform domain, perform the corresponding operations, and transform the elements back to the original domain. Other than such conveniences, the transform domain provides a deeper understanding of the related phenomena.

Take nonnegative real numbers and their logarithms. Each nonnegative real number has a corresponding logarithm. The inverse transform is also straightforward. You would take the logarithm-domain element and use it as the exponent of  $e$  (*Euler's constant*). Multiplying a bunch of real numbers is easier in the transform domain. You would simply add their logarithms. For example, multiplying gains of successive stages could be simplified by adding the decibels. MSC, anyone?

The elements of the Laplace transform are functions, such as the time-domain function  $v(t)$ . The elements of the transform domain are also functions, such as  $V(s)$ . Differentiation in time domain corresponds to multiplication by  $s$  in the transform domain. This allows differential equations in time-domain to be written as algebraic equations in the transform domain. In addition, convolution in the time domain becomes multiplication in the transform domain. These two properties alone make the Laplace transform an ideal tool for electrical engineering applications. Moreover, the Laplace transform looks very much like phasors, which allows us to interpret them with further insights.

*The topic of convolution is a worthy one, but slightly involved to be given proper attention here.*

The Laplace transform of a time-domain function  $f(t)$  is defined as,

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (7.17)$$

Most books do not specify, but I will. The units of  $s$  is the inverse of the units of  $t$ . In the original function  $f(t)$  is a function of time, then  $s$  will have the unit  $[T]^{-1}$ . The units of  $F(s)$  is the units of  $f(t)$  times  $[T]$ . For example, if we take the transform of a voltage  $v(t)$  to be  $V(s)$ , then  $V(s)$  will have units  $[V][T]$ . I suppose many books treat the Laplace transform as a mathematical formalism. Then, engineering units may seem superfluous. But for an engineer, units are of utmost importance. It allows checking results to see if they make sense. It also provides further insights into the systems.

We take advantage of the Laplace transform in phasor algebra, owing to the fact that  $j\omega$  and the transform variable  $s$  fulfill the same function, and thus are rather interchangeable in our neck of the woods.

Table 7.1: Laplace Transforms – Basic Survival Pack

Time domain $f(t)$	Transform (frequency) domain $F(s)$
$\delta(t)$ (impulse at $t = 0$ )	1
$u(t)$ (unit step)	$\frac{1}{s}$
$e^{\alpha t}$	$\frac{1}{s - \alpha}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
Operations	
$\frac{df(t)}{dt}$	$sF(s)$
$\int_0^\infty f(\tau) d\tau$	$\frac{1}{s}F(s)$

We write differential equations in the transform domain, multiplying the transform  $V(s)$  by  $s$  to capture the transform  $sV(s)$  of the derivative function  $\frac{dv(t)}{dt}$ . Reciprocally, dividing by  $s$  the transform of the time-domain function  $v(t)$  gives us the transform of the integral of  $v(t)$ .

The Laplace transform is most suited for working with transfer functions. The transfer functions of linear circuits come from ratios of functions and their various derivatives. Thus ultimately, we end up with numerator and denominator polynomials in  $s$ . Provided that you express the input in the transform domain, it becomes an algebraic task to compute the transform of the output. Once again, you must convert the output to time domain. We typically deal with simple inputs. Table (7.1 gives a minimal set of functions, their transforms, and operations.

This is only *bare-bones* information. It is a good start just to follow the rudiments we discuss here. As your journey takes you to grow as a competent engineer, I encourage you to study transforms in depth and see their many fascinating properties and applications. Good engineers are bilingual, they are equally at home thinking and speaking time-domain or frequency-domain.

## 7.7 EXPERIMENTATION AND REFLECTION

1. Regarding [Equation 7.15](#), find the cut-off frequencies of the two elemental subcircuits. Are the cut-off frequencies detectable from the transfer function?
2. Find other circuits that would implement the same transfer function (7.15). Compare the frequency response to the one given in [Figure 7.10](#). What are the advantages and disadvantages of each alternative?
3. Build a circuit that implements the transfer function

$$H(s) = \frac{s(2.1 \cdot 10^4) - 2.2 \cdot 10^7}{s^2 + s(2.0 \cdot 10^4) + 2.0 \cdot 10^6} \quad (7.18)$$

How is this related to [Equation 7.15](#)?

4. Build a circuit that implements the transfer function

$$H(s) = \frac{s(2.1 \cdot 10^4) + 2.2 \cdot 10^7}{s^2 - s(2.0 \cdot 10^4) + 2.0 \cdot 10^6} \quad (7.19)$$

How is this related to [Equation 7.15](#)?

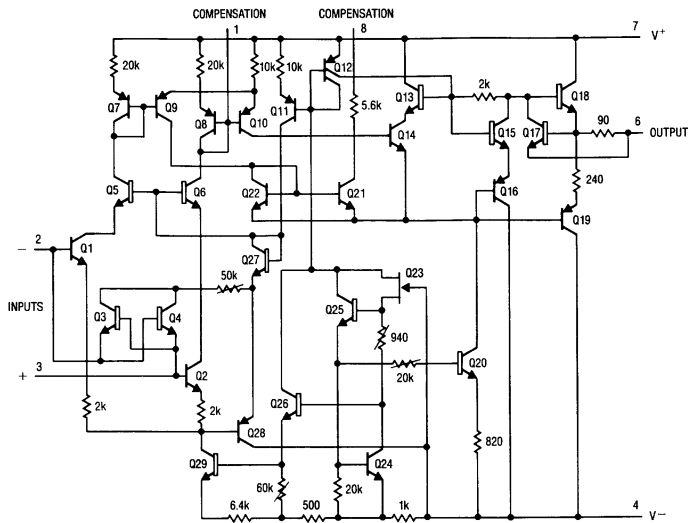
5. Suppose you wanted a transfer function whose frequency response has a double peak, like the letter 'M'. What is the minimum number of energy-storing components needed? Why?
6. Referring to the previous question, design and build such a circuit. Write its transfer function and verify that its frequency response has the prescribed shape.
7. Write the transfer function of the Sallen-Key Topology ([Figure 6.12](#)).



## Part V

# CONTROL ENGINEERING

Mind your signals and you can control the world.



Linear Technology  
LM108 Operational Amplifier, 1969

CONTROL ENGINEERING

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We somewhat get used to the way things work. I remember, as a young lad, my father used to take me to the shop. When you turn on the machines, some would rev-up for a second or two. You just get used to it, so much so that, if it takes longer or shorter to achieve normal operating speed, you suspect there is something wrong. Usually, the speed quickly builds up, but then as it gets closer and closer to the operating speed, its approach slows down. Most machines – and some people – do that. Vary rare that you see someone hit the road running. Control engineering is about setting the inputs so that the output is where you want it when you wanted, of course, within reason. This requires understanding the control side of things, but also you must know the nature of the device being controlled.

## 8.1 DEVICE MODELS

The rev-up I mentioned is often approximated by a simple system. I say simple, because we will construct a linear model. The equation defining the system dynamics will only be a first-order differential equation. Consider a constant force applied to move a mass  $m$  under friction. Let  $v(t)$  be the speed of the mass. It may be a linear motion, or rotational. If rotational, the mass  $m$  should be interpreted as its moment of inertia.

$$F = m \frac{dv(t)}{dt} + kv(t) \quad (8.1)$$

In [Equation 8.1](#) the constant  $k$  represents the friction coefficient. The equation simply says the force applied will be confronted by the inertial force and the friction force. After all, did Newton not say *for every action there is an equal and opposite reaction*? We assume that the mass is initially at rest. Let us, for good measure, put the differential equation in a standard form, complete with its boundary condition.

$$\begin{aligned}\frac{dv(t)}{dt} + \frac{k}{m}v(t) &= \frac{F}{m} \\ v(0) &= 0\end{aligned}\tag{8.2}$$

The solution to the homogeneous equation, that is, when the right-hand-side of [Equation 8.2](#) is zero, has the generic exponential form,

$$\begin{aligned}\frac{dv_h(t)}{dt} + \frac{k}{m}v_h(t) &= 0 \\ v_h(t) &= e^{-\frac{k}{m}t}\end{aligned}\tag{8.3}$$

The function  $v_h(t)$  will solve the homogeneous differential equation. We may add  $v_h(t)$  to any solution of the non-homogeneous differential equation [7.3](#), since the contribution of  $v_h(t)$  to the right-hand-side is zero. We next, in the tradition of differential equations, seek a particular solution. We note that the constant  $(F/k)$  will also solve the differential equation. Being a constant, its derivative is zero. When multiplied by  $(k/m)$  it will give us the desired right-hand-side of [Equation 6.3](#). All that remains is to combine the particular solution and the solution to the homogeneous equation to find the solution to our defining differential equation. We seek the scalar  $\lambda$  in

$$\begin{aligned}v(t) &= \lambda v_h(t) + \frac{F}{k} \\ &= \lambda e^{-\frac{k}{m}t} + \frac{F}{k}\end{aligned}\tag{8.4}$$

that satisfies the boundary condition  $v(0) = 0$ . It turns out that  $\lambda = -(k/m)$  would do the deed. We now have the specific solution to our differential equation.

$$v(t) = \frac{F}{k} \left( 1 - e^{-\frac{k}{m}t} \right)\tag{8.5}$$

I hope you knew how to solve first-order non-homogeneous differential equations, and the explicit rather pedantic exposition did not bore you. We often employ and talk about simple short-cut methods in electronics. But it is good to keep in mind that all these methods are planted in some theoretical terra firma.

Before we continue, let us inspect [Equation 8.5](#) a bit. Clearly, at  $t = 0$ ,  $v(t) = 0$ . When  $t \rightarrow \infty$  the exponential portion of the equation vanishes, and we have  $v(t) = F/k$  as  $t \rightarrow \infty$ . This also is consistent with the physics. Note that the friction force, being the product of the velocity and a coefficient is  $kv(t)$ , which in the long run matches the applied force  $F$ . Also note that the units check. Since the coefficient  $k$  has units  $[M(\text{mass})]/[T]$  (why?), the exponent is unitless, and the units of  $F/k$ , as observed, is velocity.

The response is typical of first-order systems. It asymptotically approaches steady state. The approach is referred to as *exponential decay*, since it is self similar. That is, the difference between the current value and the target value is reduced by a constant fraction per time. The same is true for radioactive decay, for example. Say, it is half per unit time. That is what gives it self similarity, and it is a dead give-away that you are dealing with a first-order system under the influence of some constant rate. If the exponent were positive, it would be *exponential growth*, such as when you have compound interest or unconstrained population growth.

How fast will the speed settle? Theoretically, it will never reach the target, but in practical terms, it will be indistinguishable from the target soon enough. The rate of decay is determined by the exponent. More specifically, by the time constant. In our case, the time constant is  $(m/k)$ . Not only does it have units of time, but at  $t = (m/k)$ , the exponential term is  $e^{-1} = 0.37$ , meaning it has made 63% of its way to the target (why?). At two time units, that is, when  $t = 2(m/k)$ , the exponential term is  $e^{-2} = 0.14$ , having made 86% of its way to the target. At three time units, the process is 95% complete. I always felt that three time-constant units is a practical cut-off value to claim the process has settled.

*The three here seems so much like the three sigma in statistics, does it not?*

## 8.2 SIMULATING DEVICE MODELS

Now that we know about analog computers and opamps, let us simulate the device explained in the previous section (8.1). Here we have a circuit ([Figure 8.1](#)) that implements the differential equation [Equa-](#)

tion 8.2. However, note that the equation is identical to one that describes charging of a capacitor through a resistor (see subcircuit a of Figure 7.4).

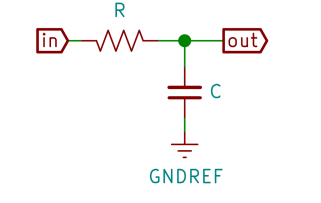


Figure 8.1: Charging a Capacitor.

Let  $v(t)$  be the voltage across the capacitor, that is the output voltage in Figure 8.1. The current through the resistor,  $(v_{in}(t) - v(t))/R$  also flows through the capacitor. Thus,

$$\frac{v_{in}(t) - v(t)}{R} = C \frac{dv(t)}{dt} \quad (8.6)$$

$$v(0) = 0$$

rearranging the terms, and adding the initial condition,

$$\frac{dv(t)}{dt} + \frac{1}{RC}v(t) = \frac{1}{RC}v_{in}(t) \quad (8.7)$$

$$v(0) = 0$$

we see that this is the same differential equation as Equation 8.2. Provided that  $v_{in}$  is a constant  $U$ , as was assumed of  $F$ , we may readily copy the solution as,

$$v(t) = U \left( 1 - e^{-\frac{t}{RC}} \right) \quad (8.8)$$

Again, we expect the same behavior. Once the input voltage  $U$  is applied to an empty capacitor, the voltage  $v(t)$  will rise asymptotically

to reach the supply voltage  $U$ . The time constant is  $RC$ . And once again, at 1, 2, and 3 time constants, the voltage will have reached 63%, 86%, and 95% of its target value of  $U$ .

Build and observe the circuit, or at least simulate it (Figure 8.2). It is best to select a rather large time constant so things unfold slowly. A  $100\mu\text{F}$  capacitor charged by a  $10\text{k}$  resistor would give us a time constant of 1 second.

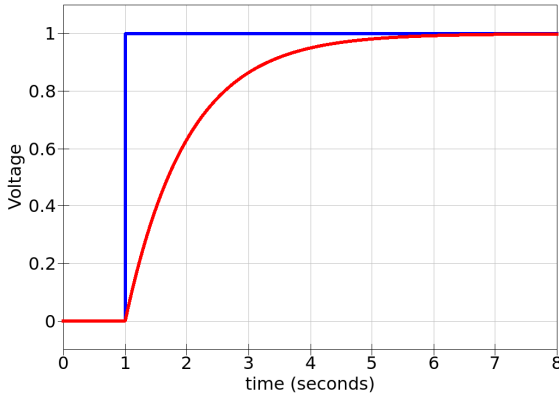


Figure 8.2: Voltage on a Capacitor being Charged.  
(blue: excitation, red: capacitor voltage)

The blue line is the excitation, a 1V signal applied at time  $t = 1\text{s}$ . The red line shows the voltage on the capacitor. Can you spot the magical 63%, 86%, and 95% progress?

This simulation is more than just observing the similarities between damped movement under a constant force and charging a capacitor. The simple resistor-capacitor circuit will serve to model a first-order device. Control engineers do not use the term *device* but rather *plant*. The term comes from chemical engineering, where control engineers wanted to carefully control the chemical processes by adjusting flows, mix rates, evaporation, heating, cooling, etc. So, to the chemical engineer, the thing to control is *the plant*.

### 8.3 PROCESS CONTROL

For machines revving up, a short start time is insignificant. But sometimes you may want to do something to speed up the process. A good

example is when you are driving, you most likely hit the gas a bit more to accelerate, like many do. Then ease off the gas to cruise. The initial jolt overcomes the inertia – and the car has some inertia. I remember cleaning the carburetor “acceleration boost” pumps in the 70s. These were little bellows that pumped a metered amount of extra fuel if you hit the accelerator in a hurry. Modern engine management systems do it too.

*Recently, there is a bit of chatter regarding acceleration boost modifications on Teslas.*

So, what can we do to charge the capacitor faster? What may we do to provide this *acceleration boost*? This, of course is an allegory. What we really want is to speed up a process reaching its steady state. A boost may also apply to charging capacitors (our first-order model of a plant). We start with a higher voltage and then lower the voltage to the target  $U$  as the capacitor charges. With the larger-than- $U$  initial voltage, that is the *boost*, we force more charge into the capacitor sooner.

Here is the key idea: transfer functions become quite handy when we study such control. In fact, transfer functions are the indispensable tool of control engineering. Here is a simple, first-order linear approach that would come natural to you. Apply a voltage proportional to the difference between the capacitor voltage and the target voltage. That is, let  $e(t) = U - v(t)$  be the error at time  $t$ . Let the voltage applied be a positive multiple of the error. As the error nears zero, so does the control voltage drop.

Take a look at [Figure 8.3](#). The output is sensed and fed back by the inverting opamp A2, thus effectively multiplying the output by  $-1$ . This is added to the input signal by the summing opamp A1. This opamp finds the difference between the input and the output. That difference is the error. The summing opamp A1 also multiplies the error by 100. This is the control signal used to excite the “plant”.

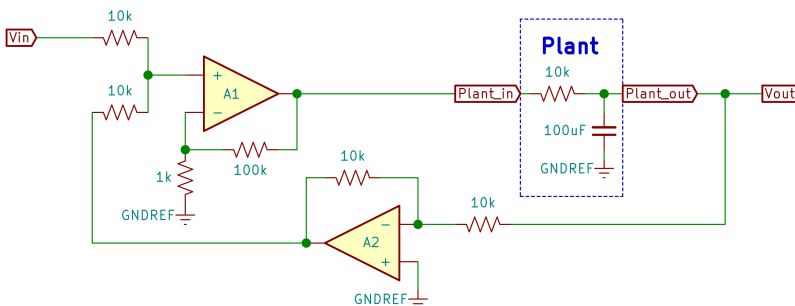


Figure 8.3: Control Circuit (Excitation is Proportional to Error).

The only parameter of the proportional controller is its gain, here  $K = 100$ . Depending on the plant, the controller is *tuned* by selecting the appropriate gain. Too little gain, and the settle time will be long. Too much, and depending on the plant, the controller may overshoot or be unstable. We will briefly revisit the stability issue in [Section 8.6](#).

When the target input  $U = 1\text{V}$  is applied, the response of the circuit is observed as given in [Figure 8.4](#). Compare this to [Figure 8.2](#). The blue line is the target input ( $U$ ), the green the excitation, or the output of the control signal generated by the summer-multiplier. The red line show the charge on the capacitor, or if you like, the state of the plant. The time constant is 1 s but the capacitor here charges in less than half-a-second. Of course, this is due to the fact that there is considerable boost in the beginning. In fact, the excitation saturates at 15V, which is the supply voltage of the opamp. So, technically speaking, our controller is not linear, since to be linear, the control voltage at the beginning would have to be 100V (why?).

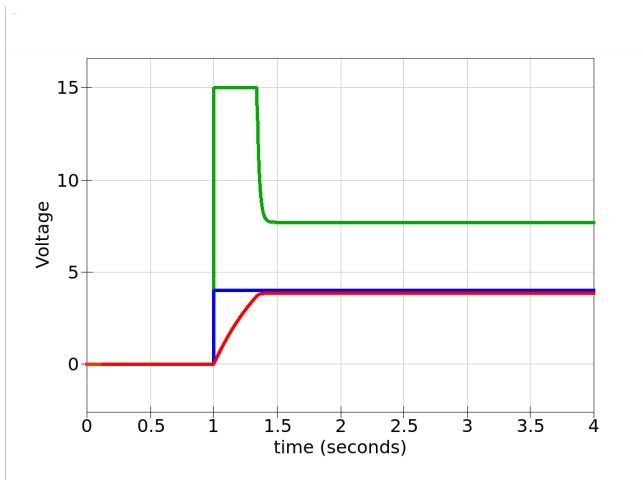


Figure 8.4: Control Response (Excitation is Proportional to Error).  
(blue: excitation, green: controller output, red: capacitor voltage)

Did you notice that the capacitor voltage never reached the input? It is just a little less than the target. Why do you think this is? Instinctively, you may say that at steady state, the output of the control circuit must be the same as  $U$ . But this would mean that there must be an error of  $(U - V)/G$ , where  $G$  is the gain of the opamp. So the output never reaches the target and there is always a little error, appropriately called the *steady-state error*.



We can gain a better understanding of the *system*, that is, the controller plus the plant, if we use transfer functions. You see, the output of the controller is fed into the plant, while the output of the plant is fed back into the controller. From a transfer-function viewpoint, we have strung along subsystems. So the behavior of the entire system may now be evaluated *algebraically* by our transfer-function approach. This is perhaps the primary reason transfer functions are so prevalent in electrical engineering. It provides a convenient pathway to analytical thinking.

*The adjective "analytical" literally means "understand the whole by breaking into parts and studying the parts."*

Let us consider a subsystem with the transfer  $H(s)$  as shown in Figure 8.5. The output  $V(s)$  is fed back through the subsystem whose transfer function is  $G(s)$ . The input  $U(s)$  then added to the feedback signal. All signals and transfer functions are expressed in the (Laplace) transform domain, which allows us to use algebraic equations to describe the net input-output characteristics of the system.

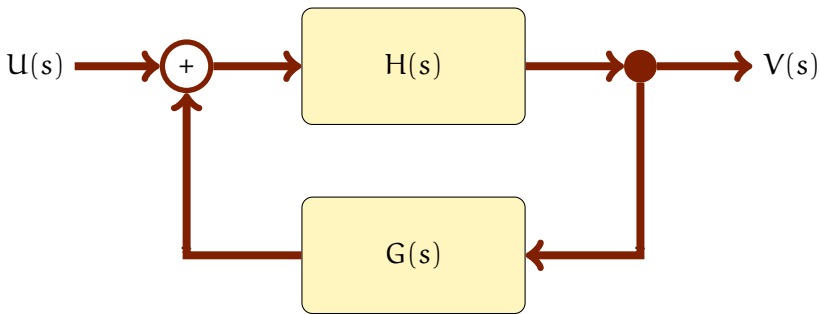


Figure 8.5: Closed-Loop Control.

We have,

$$V(s) = (V(s)G(s) + U(s))H(s) \quad (8.9)$$

Combining the terms  $V(s)$ , we get,

$$V(s) = U(s) \frac{H(s)}{1 - G(s)H(s)} \quad (8.10)$$

or,

$$\frac{V(s)}{U(s)} = \frac{H(s)}{1 - G(s)H(s)} \quad (8.11)$$

This means that we can replace the original system, the transfer function  $H(s)$  along with that of the feedback subsystem  $G(s)$ , with a single transfer function  $Q(s)$ .

$$Q(s) = \frac{H(s)}{1 - G(s)H(s)} \quad (8.12)$$

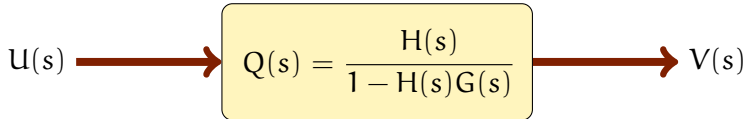


Figure 8.6: Equivalent System.

Now let us apply this to our example. In our case, the transfer function  $H(s)$  has two components, the gain ( $K = 100$ ) and the resistor-capacitor circuit from [Figure 8.1](#), as discussed in [Section 7.3](#) (see [Figure 7.4](#)). The controller is actually just the gain. The plant is the resistor-capacitor circuit.

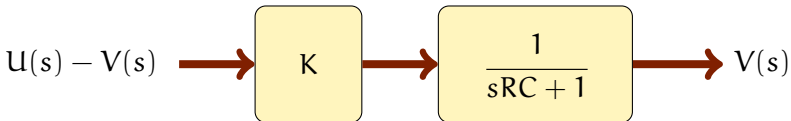


Figure 8.7: The Controller and the Plant.

The transfer function  $H(s)$  is thus,  $\frac{K}{sRC + 1}$ . Now we can apply the expression we developed for the feedback system ([Equation 8.12](#)).

$$H(s) = \frac{K}{sRC + 1} \quad (8.13)$$

The feedback is simply the negative of the output,  $G(s) = -1$ . Thus,

$$\begin{aligned}
 Q(s) &= \frac{H(s)}{1 + G(s)H(s)} \\
 &= \frac{\frac{K}{sRC + 1}}{1 + \frac{K}{sRC + 1}} \\
 &= \frac{K}{sRC + (K + 1)}
 \end{aligned} \tag{8.14}$$

The result seems awfully close to the good-old transfer function we have for the simple resistor-capacitor voltage divider. In fact, by tweaking the expression just a bit, we get

$$\begin{aligned}
 Q(s) &= \frac{K}{sRC + (K + 1)} \\
 &= \left( \frac{K}{K + 1} \right) \frac{1}{s \left( \frac{RC}{K + 1} \right) + 1}
 \end{aligned} \tag{8.15}$$

Now, the transfer function is just like the one where we were charging the capacitor. Except for two things. The time constant is no longer  $RC$ , but  $\frac{RC}{K + 1}$ , and there is a gain of  $\frac{K}{K + 1}$ . That the time constant is reduced means that the circuit will act much faster, and charge the capacitor much sooner. The gain is actually less than unity, not by much, but nonetheless, a little. Without the controller circuit, and with a time constant of 1 second, the capacitor was practically charged in 3 seconds, if you recall discussion about the 95% in 3 time constants (see [Section 8.1](#)). Now that we have the controller with  $K = 100$  we do expect the time constant to be about 100 times shorter, and thus the capacitor to charge in about 30 mSec. Looking at [Figure 8.5](#), we see that this is not the case. Why? Well, because our controller could not provide the 100V at the onset, but saturated at its supply voltage of 15V. So, we settle with a slower response time. Still, the opamp did the best it could muster and the response is much faster than without the controller.

The gain (actually, the attenuation)  $\frac{K}{K+1}$  explains the steady-state error. In fact we could have calculated this without the transfer function. Let  $U$  and  $v$  be the input and output voltages at steady state. Then,  $v = K(U - v)$  which gives us the same result,  $\frac{v}{U} = \frac{K}{K+1}$ .

#### 8.4 REMOVING THE STEADY-STATE ERROR

Our controller uses the amplified the error signal to excite the plant. Several extensions are possible. Typically, not only the error, but the integral and the derivative of the error may be used as the control signal. Specifically, the integral of the error is useful, as its reduction removes steady-state errors when the input is a constant (step input), or at least, when the input changes infrequently.

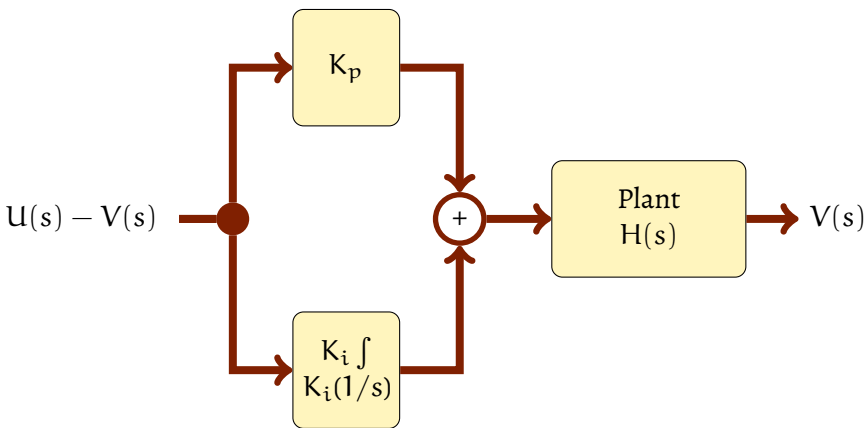


Figure 8.8: The Proportional-Integral Controller and the Plant.  
( $K_p$  is the gain of the proportional controller and  
 $K_i$  is the gain of the integral controller)

The controller is similar to proportional control. We still generate an error signal, which is the difference between the input and the output. This signal is now fed into two subcircuits. As before, one generates a proportional control component of the excitation. In addition, the same error signal is sent to an integrator. In [Figure 8.8](#) we show the branch that integrates the error signal with the  $\int$  sign as well as with  $1/s$ . This is because integrating the signal in the time domain is equivalent to dividing the transform<sup>1</sup> domain function by its variable  $s$ . The

<sup>1</sup> Laplace transform, of course

integral error also has a gain, here  $K_i$ . The outputs of the proportional and integral errors are added and used as the plant-driving excitation. Let us apply the concept to our circuit. Again, we use a unit step input to observe how quickly steady state is reached. The plant model is the same first-order system with a time constant of 1 Sec, equivalent to charging a  $100\mu\text{F}$  capacitor through a  $10\text{k}$  resistor.

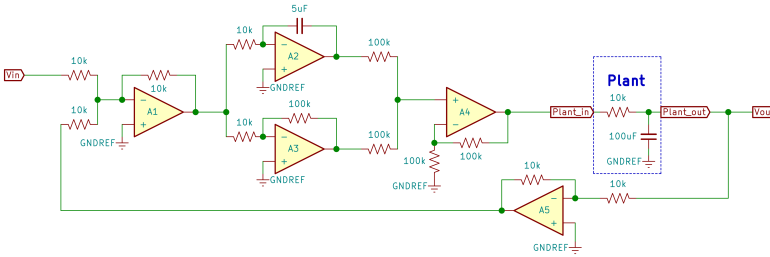


Figure 8.9: Proportional and Integral Control Circuit.

You probably recognized the various sections of [Figure 8.9](#). Once again, the output voltage is inverted ( $A5$ ) and added to the input by a summer ( $A1$ ). This is the error voltage that will now be used by the proportional and integral amplifiers. Opamp  $A2$  integrates the error voltage, and  $A3$  amplifies it ( $K_p = 10$ ). Note that the integrator  $A2$  also has a gain of  $K_i = 200$  (why?). It integrates and amplifies – nice, is it not? The summer amplifier ( $A1$ ) is configured as an inverting amplifier. But the integrator  $A2$  and the amplifier  $A3$  are also in the inverting configuration. So, while the negative of the error is being integrated and amplified, the outputs at  $A2$  and  $A3$  are once again sign-corrected. The following stage adds the outputs of  $A2$  and  $A3$ . Now that the signals are sign-correct, we made sure that the summer  $A4$  is in the non-inverting configuration.

There is a bit of subtlety here – did you catch it? The output of the summer amplifier is actually the average of the two input signals. It is good that the two input resistors to  $A4$  are the same. Otherwise, it would not be a simple average. With identical resistors, the voltage at the non-inverting input is the midpoint between the two signals to be added. So, the output of  $A4$  is the sum divided by 2. As an added bonus,  $A4$  also has the capability to amplify the sum of signals. Here, the gain is set to 2, which makes  $A4$  a true summer. You may choose to increase the gain of this stage by increasing the value of the feedback resistor. The gain may be reduced all the way to unity by configuring  $A4$  like a voltage follower.

We kept the same plant model for now. We may change the plant model in further experiments, but the basic circuit configuration would serve us fine. The controller now has two parameters. The gain of the proportional controller and the gain of the integral controller. With these, we may tune the controller to our needs. First, though, we will try to eliminate the steady-state error of the proportional-only controller.

The proportional controller with gain  $K = 10$  gives an almost 10% steady-state error. We investigated this phenomenon in the preceding section (Section 8.3). As seen in Figure 8.10, the steady-state error diminishes when the controller also uses the integral of the error signal.

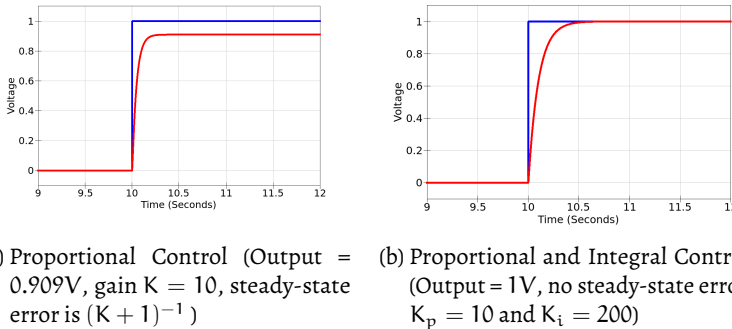


Figure 8.10: Proportional and Integral Control (Unit step input 1V).

The unit step input is applied at time  $t = 10$  Sec. Once again, we see that the time to settle to steady state is reduced from approximately 3 seconds with no controller to about half a second with the controller.

## 8.5 A SECOND-ORDER PLANT

While the preceding sections provide insights into control, the simplicity of the plant somewhat hides many design concerns. For starters, let us consider plants with more character. After all, charging a capacitor translates into a rather mundane transfer function.

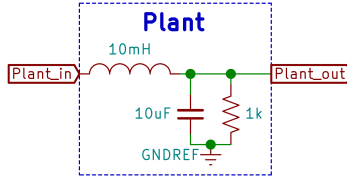


Figure 8.11: A Second-Order Plant Model.

The tank circuit (8.11) would make an interesting plant model. We know that this circuit would have a resonance frequency proportional to  $\frac{1}{\sqrt{LC}}$ . As  $LC$  is in the order of  $10^{-7}$ , we would estimate, order-of-magnitude-wise, the resonance frequency to be around 1kHz. Using the proportional control as in Figure 8.3, with a gain of  $K = 10$  we observe the response of the controller to a unit step input at time  $t = 1$  Sec. The system quickly settles, but not without ringing like a bell. The steady-state error seems like the case with the first-order plant model.

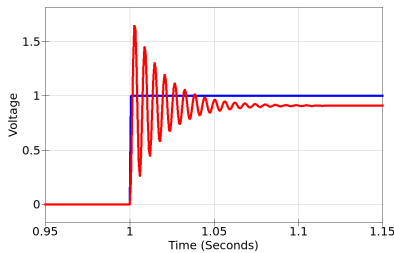


Figure 8.12: A Second-Order Plant Model with a Proportional Controller.

Next, we try the integral controller as given in Figure 8.9, but without the proportional controller. The response is swift and without the ringing we see with proportional control. If we use both the proportional and the integral controller, it seems we must tune the circuit so that the proportional controller gain is low. After a certain threshold, the proportional controller induces the characteristic high-frequency ringing we have observed.

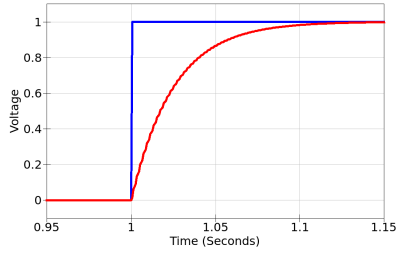


Figure 8.13: A Second-Order Plant Model with an Integral Controller.

The proper way to analyze the effect of the controller is to construct the transfer function of the entire system: the plant, the controller, and the feedback portion of the circuit. We had developed all the needed tools in [Section 7.4](#), [Section 8.3](#), and [Section 8.4](#) to do this. The controller has a transfer function,

$$C(s) = K_p + \frac{K_i}{s} \quad (8.16)$$

where  $K_p$  is the gain of the proportional controller and  $K_i$  is the gain of the integral controller. The term  $1/s$  corresponds to integrating the signal. The plant is a tank circuit built with a capacitor, an inductor, and a resistor. Let  $C$ ,  $L$ , and  $R$  be the values of these components. The transfer function is computed as,

$$P(s) = \frac{R}{s^2 RCL + sL + R} \quad (8.17)$$

The feedback portion of the controller is simply an inverter, and thus its transfer function is  $G(s) = -1$ . The transfer function of the feedback system was developed in [Equation 8.12](#). The controller and the plant combination has the transfer function,



$$\begin{aligned}
H(s) &= C(s)P(s) \\
&= \left( K_p + \frac{K_i}{s} \right) \frac{R}{s^2 RCL + sL + R} \\
&= \frac{K_p R}{s^2 RCL + sL + R} + \frac{K_i R}{s^3 RCL + s^2 L + sR} \\
&= \frac{sK_p R + K_i R}{s^3 RCL + s^2 L + sR}
\end{aligned} \tag{8.18}$$

The equivalent transfer function for the entire system is then computed as developed by [Equation 8.12](#).

$$\begin{aligned}
Q(s) &= \frac{H(s)}{1 + H(s)} \\
&= \frac{\frac{sK_p R + K_i R}{s^3 RCL + s^2 L + sR}}{1 + \frac{sK_p R + K_i R}{s^3 RCL + s^2 L + sR}} \\
&= \frac{sK_p R + K_i R}{(sK_p R + K_i R) + (s^3 RCL + s^2 L + sR)} \\
&= \frac{sK_p R + K_i R}{s^3 RCL + s^2 L + sR(K_p + 1) + K_i R}
\end{aligned} \tag{8.19}$$

Now that we have an expression for the transfer function of the entire system, we may study its behavior and select the parameters, namely  $K_p$  and  $K_i$  to *tune* the controller to best suit the plant whose characteristics are determined by the values of  $R$ ,  $C$ , and  $L$ .

Does the fact that our transfer function has a denominator polynomial of degree 3 surprise you? It indicates that the entire system is a

third-order system. Well, it should not. The plant is already a second-order system. Coupled with an integrator increments its order once more.

One last thing: what about the units? Look at the coefficients of the polynomials. The reciprocal of time is frequency. Recall that  $s = j\omega$  and thus has units  $[T]^{-1}$ . The gain  $K_p$  of the proportional controller is a unitless constant, since it is the ratio of the output voltage to the input voltage of that subcircuit. The gain  $K_i$  of the integral controller has units  $[T]^{-1}$ . Why? Well because the input to the integrator is voltage but the output is voltage integrated over time. So the output has units  $[V][T]$ . The gain is the ratio of the output to the input. To make the subcircuit consistent the gain must have units  $[T]^{-1}$ . This makes the units of the numerator terms consistent, both terms have units  $[R][T]^{-1}$ . The denominator of  $Q(s)$  has four terms. The last two terms,  $sR(K_p + 1) + K_i R$  look just like the numerator, and thus have the same units,  $[R][T]^{-1}$ . What about the first two terms in the denominator? The impedance of the inductor is  $sL$  which has the same units  $[R]$ . So,  $s^2L$  has the same units as  $sR$ , and thus is also consistent with the units of the numerator. Finally, the first term,  $s^3RLC$  may be seen as the product of  $sRC$  and  $s^2L$ . We already found that latter term has units  $[R]/[T]$ , but what about the term  $sRC$ ? Since  $RC$  is a time constant, the term  $sRC$  is unitless. Both the numerator and the denominator have units  $[R][T]^{-1}$ . Such checks is useful when you do some calculation, It is good to go over your units as part of a *design rule check* step – just to make sure.

*Once again, this is why the transform domain is referred to as the frequency domain. Its variable  $s$  has units  $[T]^{-1}$  (frequency).*

## 8.6 TUNING CONTROLLERS

The preceding approach and methodology would work for almost any control task. We work with transfer functions, combine the controller and the plant, and then the feedback subsystem. Our feedback is simply an inversion, but there may be other, more sophisticated ways of designing controllers with fancier feedback circuits. Ultimately, the transfer function of the entire system becomes *parameterized* as we see here. That is, the specifics of the transfer function now relies on the choice of parameters such as the individual gains. By selecting appropriate parameters, we tune the controller. Typically, there is an objective function and some constraints. For example, we may wish to minimize the settle time for a given input, while keeping the overshoot below a certain limit. High on our mind should be the stability of

the system. Note that we encounter oscillatory responses if the zeros or poles are complex. Keeping them real eliminates any oscillatory behavior. Moreover, the real components of the complex roots of the denominator or numerator polynomial would guarantee that the signals settle and not grow boundlessly.

Engineers sometimes view tuning such circuits as *relocating the poles and zeros* of the transfer function. Studying how (in which direction) the poles or zeros move as you change the system control parameters is a worthy topic. You could in principle, write the transfer function and compute the poles and zeros by some engineering software. Then plot these as you loop through different parameter values. This would be a much insightful exercise. Doing so analytically, as one may appreciate, is possible, but somehow best left to a subsequent review of the topic.

## 8.7 EXPERIMENTATION AND REFLECTION

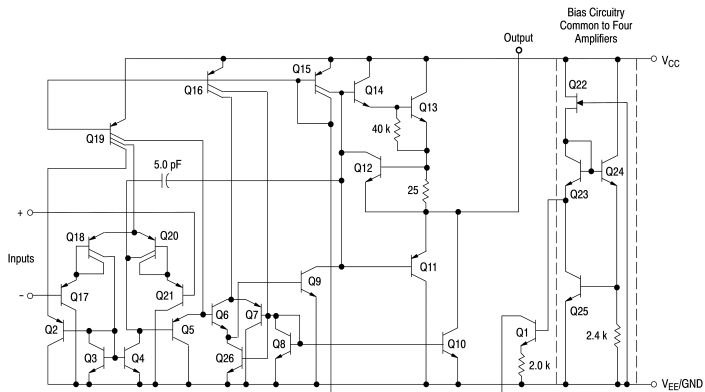
1. Adjust the gain of the controller in [Figure 8.3](#) so that the opamp does not saturate. Observe the time constant of the controlled system and verify that our computations ([Equation 8.14](#)) hold.
2. Reduce the gain of the controller in [Figure 8.3](#) and observe the response times. That would happen if the gain is less than unity?
3. Try different plant models in [Figure 8.3](#) and observe the response times over a range of gains, say from  $K = 1$  to 1000.
4. Change the gain of the proportional and the integral components of the control signal in [Figure 8.9](#). What are the affects of these parameters?
5. Note that when integral control is used to control a first-order plant, the aggregate controller-plant system becomes a second-order system. Second-order systems are generally capable of oscillatory behavior. How would you change the controller parameters to induce or suppress oscillatory behavior?
6. Review the proportional-integral controller of [Figure 8.9](#) and the second-order plant model of [Figure 8.II](#), along with the transfer function as computed by [Equation 8.19](#). Using the transfer function, compute the frequency of the oscillations before the controller settles. Compare this value to the observed frequency. How do the controller parameters affect the frequency.

7. Again, referring to [Equation 8.18](#), explain why proportional control seems to promote oscillations, while integral control seems to suppress them.
8. Compute the poles and zeros of the transfer functions developed in this Chapter. Try different parameter values and plot the values of the poles and zeros as a function of parameter values.
9. Considering the poles and zeros of the transfer functions developed in this Chapter, is it possible that a pole and a zero would coincide? If so, what does this say about the transfer function and the system?
10. Can you think of non-linear control schemes? Design and experiment with these and see if they have any advantages over linear controllers.

## Part VI

### OPAMP APPLICATIONS

Workhorse circuits, power circuits, fun circuits, clever circuits, wickedly clever circuits.



National Semiconductor  
LM324 Quad Operational Amplifier, 1972

## COLORFUL OPAMP CIRCUITS

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Now that we have a grasp of the basics, let us take a look at a few intermediate opamp circuits. Our objective is not to provide a complete set of rather well-known applications, but to present enough different circuits to show the versatility of the device and the richness of its application domains. While we design our circuits, we continue to assume that our opamps are *ideal* devices, that is, many of their parameters are extreme (see [Table 2.1](#)).

These assumptions simplify design efforts. However, when we think about practical aspects of the design, we must pay attention to the limitations of the opamp. A similar case was encountered in [Section 8.3](#) when the proportional controller output was saturated at the supply voltage, and hence, the settle time was longer than ideal. Good engineer practices promote the achievement of goals through robust design, rather than the use of exotic components. For example, if noise is important in your design, then composing the circuit, selecting the component placements and the tracks, etc. must be on your mind. A haphazard design, but one that uses a special low-noise opamp would be frowned upon. Not that it may not work, but rather it would seem like sloppy professionalism. After all, one may submit a rather Kantian[6] claim that doing so unnecessarily wastes resources, and thus is harmful to society and to the environment.

*I have often wondered about this. Should these concerns not be included in engineering ethics?*

We start with a tribute to many an opamp whose lives are dedicated to serving as *building blocks*. There so many such applications that a complete survey would require a treatise of orders of magnitude more voluminous than this book. If you seek a large set of different applications, I would recommend how I delved into the subject: by reading application notes of chip manufacturers. A nice collection is by Texas Instruments, which is now also available as a book [7]. Good application notes are available from many other manufacturers, Analog Devices, Linear Technology, National Semiconductor, Maxim Integrated, etc. Companies seem to change, merged or acquired, over time, but their application notes often remain available. Before the Internet, I vividly recall the 4-inch thick data-book from National Semiconductor, and

the many a projects I designed using ideas from it. If nowhere else, you may find these old books preserved on [www.archive.org](http://www.archive.org).

## 9.1 SOME WORKHORSE OPAMP CIRCUITS

Many opamps are used as voltage or current sources, especially in instrumentation applications. Consider the following voltage-controlled current source circuit.

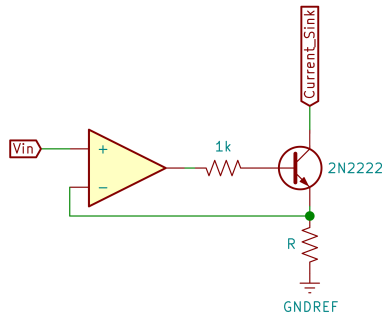


Figure 9.1: A Current Sink.

The circuit works as a current sink. The current that flows through the circuit is determined by the input voltage  $V_{in}$  and the value of the resistor  $R$ . The current is  $I = \frac{V_{in}}{R}$ . The circuit uses a common silicon NPN transistor, one that I have reached out to for many such routine tasks. We used to buy them in large bags and use the 2N2222 as our default small-signal transistor. The 1k at the base is a current-limiting resistor, mostly for the protection of the transistor. When  $R = 1k$  for example, you get a 1 mA per volt sink. This circuit is just a generic way of looking at current sinks. Many extensions present themselves. You may, for instance, set the input voltage by a zener diode and construct a fixed current sink. Or you may use a Darlington transistor to sink higher currents. You may use a PNP transistor and build a current source, rather than a current sink. Typically, the current sink would be a subcircuit of a larger system. As you gain experience and scan circuit diagrams, once again, you come across such circuits and they register as building blocks rather than individual components.

Not all opamp circuits are linear circuits. The following is known as the *ideal rectifier*. The operating principle should be fairly easy to spot

once you notice that  $V_{out}$  is actually the same as the voltage at the noninverting input. The two inputs of the opamp would be essentially at the same voltage, when  $V_{in}$  is positive. Then the diode is forward biased, with current flowing through the load resistor. That would mean that the output voltage is the same as the input voltage. However, when the input voltage is negative (below the ground reference), then the diode is reverse biased, with no current flowing through it. The voltage at the output would be the same as the ground reference. In this case, of course, the two inputs of the opamp would not be the same. The opamp output would be saturated, close to the negative supply.

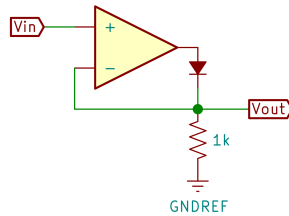


Figure 9.2: Ideal Rectifier.

It is the art of engineering, once articulated, to put such basic building-block circuits in your toolbox, only to be revisited, extended, and re-purposed. A good extension is the so-called *max* circuit.

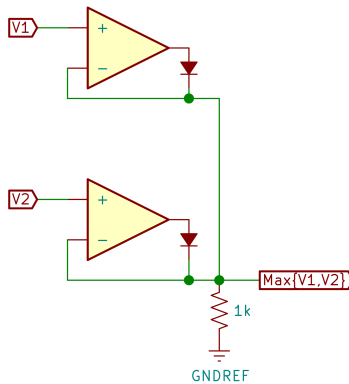


Figure 9.3: Maximum of the Inputs.

I am sure you see the *family resemblance*[9] to Figure 9.2 here. Each opamp operates as an ideal rectifier. The input voltage that is higher



not only drives the output, but since this voltage is greater than the other input, it forces the other opamp to saturate at the negative supply voltage. Here is another nice extension. This one is known as the peak detector. Actually, it does not *detect* the peak, but the locks on to and outputs the peak voltage.

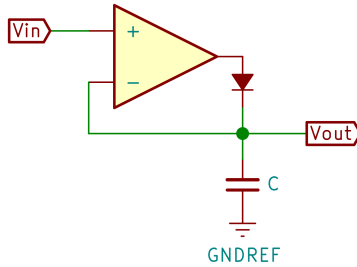


Figure 9.4: Peak Voltage Detector.

Once again, it is easy to see the basic rectifier circuit hidden in the peak detector. Of course, over time the capacitor will leak and the peak voltage will drop. In practical circuits, it probably would suffice to keep the peak voltage for a limited time, anyway.

Now let us switch our attention to a different class of circuits. We discussed active filters in [Part III](#). If you are designing filters for audio applications, you would naturally want to keep distortion at a minimum. You would be careful not to saturate the opamps, otherwise the clippings at the extremes would cause unwanted distortion. But some actually like distortion. It is legendary that early electric guitar players, due to the inherent quality of the amplifiers, have used distortion as a musical element. How would you create such distortion? Here is one popular approach.

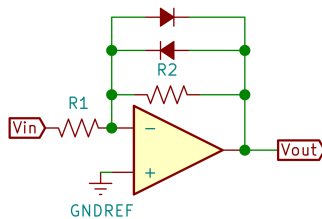


Figure 9.5: Simple Fuzz.

The resistors form the usual amplifier. However, silicon diodes have a forward voltage drop of about 0.7V. This limits the output voltage range, effectively clipping the signals. You may make one of the resistors a potentiometer and change where the signals will be clipped. The severity of clippings, and thus the distortion, is often controlled by a potentiometer labeled *fuzz*. Then, another output stage with its own potentiometers independently may set the volume and the tone.

*I confess, this type of fuzz does not yield the best sound.*

A set of outputs is given in Figure 9.6 with the input set to the high A note (880 Hz). Even though the shape of the output varies just a little, the ear will perceive these at different levels of fuzziness. The higher the gain, the more rectangular the output signal. The rises and drops are more vertical when the gain is higher. The ear catches all this because the harmonics of the distorted signals change quite a bit when their shapes so change. Note that we did not use an ideal diode, but one that is more realistic (1N914). The rounded corners of the clippings attest to the fact that the forward voltage drop changes with increasing current through the diodes.

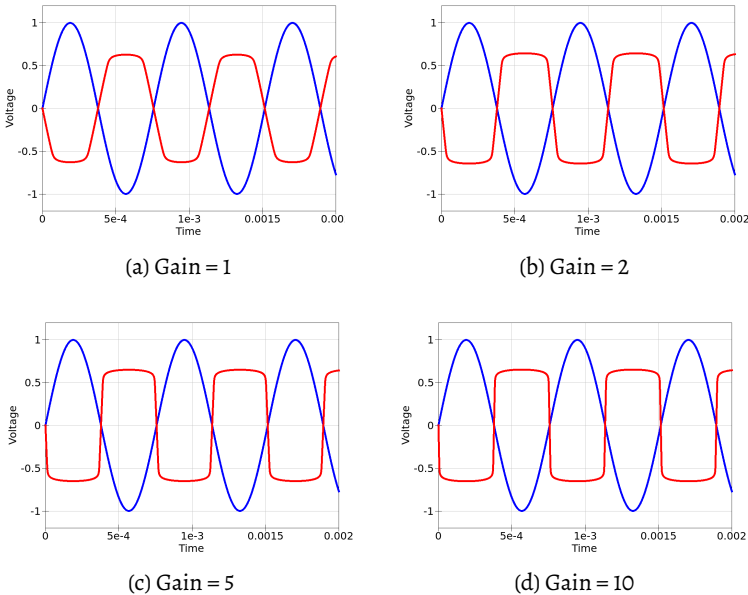


Figure 9.6: The Effect of the *Fuzz* Setting on Clippings.  
(blue: input signal, red: clipped output signal)

I must, however, say that opamp-based fuzz pedals do not quite live up to audiophile expectations. Pedals made with individual transistors

are preferred, partially due to the sound that they produce. Also perhaps, as the signal passes through fewer transistors the signal remains somewhat raw and authentic.

Speaking of periodic signals, opamps make nice astable multivibrators. Consider the basic idea given in [Figure 9.7](#).

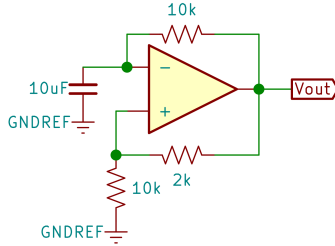


Figure 9.7: An Astable Multivibrator.

Do you recognize the hysteresis – the use of positive feedback? If not, revisit [Section 1.10](#). Now, do you also recognize the voltage divider? The 10k resistor and the 10 $\mu$ F capacitor form our trusty old voltage divider. As the capacitor is charged, the voltage at the inverting input raises until it exceeds the voltage at the noninverting input. Then, the output switches, and the capacitor is discharged until its voltage drops below the voltage at the noninverting input. The noninverting input, of course, is fed by the 2k – 10k voltage divider. As the output swings between the supply voltages, the voltage at the noninverting input switches between  $V_+ = \frac{10}{12}V_s$  and  $V_+ = -\frac{10}{12}V_s$ , where  $V_s$  is the supply voltage. The fraction  $\frac{10}{12}$  is close to our magic 86%. Now, as a voltage divider, the time constant is 10 $\mu$ F $\cdot$ 10k, or 10 mS. It will take around two time constants to charge the capacitor to 86% of the supply. So we should expect a relatively low frequency at the output, in the 1Hz to 10Hz range.

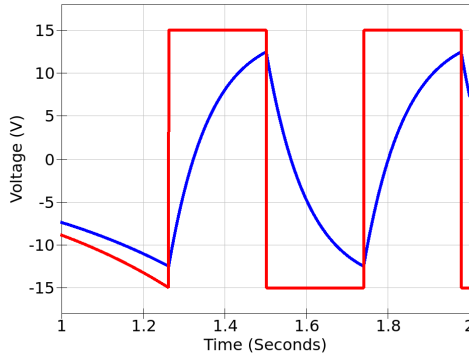


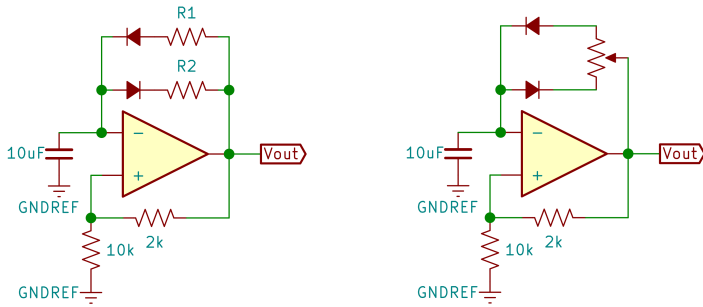
Figure 9.8: The Output of the Astable Multivibrator.  
(red: output, blue: voltage on the capacitor)

The output is a symmetrical square wave. It is symmetrical, because the charge time and the discharge time of the capacitor are the same.

This is a good place to combine the things we know to create new things. This is the spirit of synthesis – creation from combination of known parts. It is the opposite of analysis. But not the same as creativity that produces a completely new thing. Most engineering design is synthesis, not absolutely novel innovations. Those often come from basic sciences, anyway. So, let us say that we do not want a symmetrical output, but one that has a certain *duty cycle*. The term “duty cycle” is the fraction of time the square-wave signal is at a high voltage. By the way, we say square wave, but we really know that the shape is more like a rectangle. You may, of course play with the units of time and voltage to make the geometry on the graph look exactly like squares.

Duty cycle is a nice feature. You may use it for *pulse-width modulation*, essentially changing the duty cycle to adjust the energy output. It is also commonly used in servo motors, the type radio controlled models use. There, the pulse width determines the position of the servo. Consider the following extension to Figure 9.7. This is a rather obvious extension. We use diodes to channel the charging and discharging currents to go through dissimilar resistors, thereby changing the charge and discharge times.

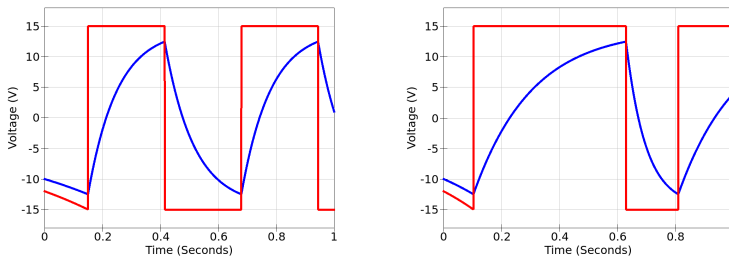
*I was so proud of myself when I thought of putting a trimmer (small potentiometer you set with a screwdriver) to set the duty cycle.*



(a) Duty cycle determined by  $R_1$  and  $R_2$     (b) Duty cycle determined by the potentiometer

Figure 9.9: Asymmetrical Astable Multivibrators.

Let us build Figure 9.9a first with  $R_1 = R_2 = 10k$  and then with  $R_1 = 20k$  and  $R_2 = 6.8k$ . It would be best if we could find a 6666 Ohm resistor to keep the periods of the two signals the same. The standard value closest to our needs is 6.8k. The standard values are selected for a reason.



(a) Duty cycle is 50%    (b) Duty cycle is  $\approx 75\%$

Figure 9.10: The Output of Asymmetrical Astable Multivibrators.  
(blue: capacitor voltage, red: output voltage)

Since you know well how to compute the time that the voltage on a charging capacitor reaches a given fraction of the supply voltage, you should have no trouble computing resistor values that would result in a given frequency and duty cycle for your multivibrators.

There are many many more possible applications of opamps, but for an initial exposure, let us leave the fun circuits and look at some quite ingenious designs.

## 9.2 POWER CIRCUITS

Sometimes you want to drive heavy loads. The output of an opamp is typically around a few hundred mW (milli-Watts). You could simply add a *push-pull* stage to your opamp. This configuration is called “push-pull” because the NPN transistor *pushes* current to the load (sources), and the PNP *pulls* (sinks). Together they drive the load, both sourcing and sinking current.

*Named after James Watt (1736-1819), of course, the Scottish inventor-engineer of the steam engine fame*

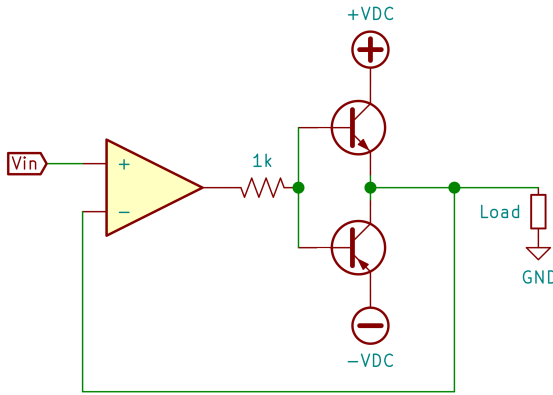


Figure 9.11: Power Boost.

If your application is to drive relays, or motors, etc. then [Figure 9.11](#) would probably suffice. The transistors may even be Darlington transistors to provide more output current. When the output is at ground reference, both transistors are off, so the quiescent current is practically non-existent. However, it will take some time for the transistors to turn on, so the circuit will suffer from *cross-over distortion*. Cross-over distortion is easily eliminated by adding a bit of quiescent current to keep the transistors at the verge of turning on ([Figure 9.12](#)).

You may be wondering why I did not specify the resistors values. Well, those depend on a few things. The choice of the transistors, probably, would have the greatest say in how to bias the output stage. Here is one ([Figure 9.13](#)) that would work up to 80V and 1.5A. It uses the popular complimentary pair of medium-power transistors BD139 – BD140.

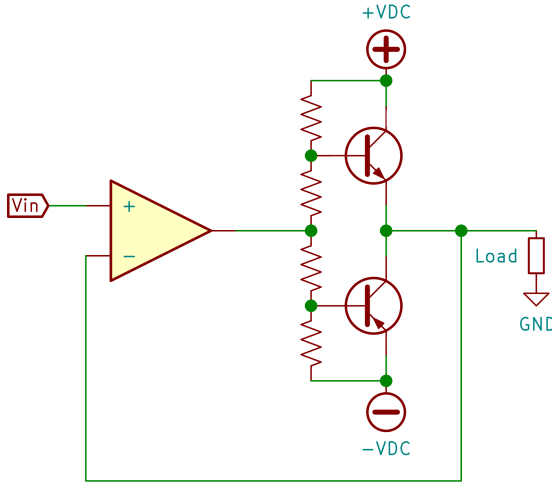


Figure 9.12: Power Boost with no Cross-Over Distortion.

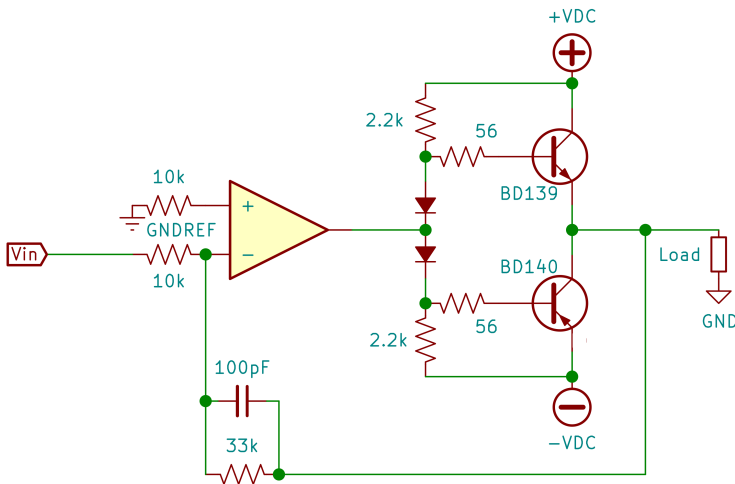


Figure 9.13: A Practical Power Boost Stage.  
(Use small signal silicon diodes such as 1N914)

The opamp is configured as an inverting amplifier with gain 3.3. The small capacitor in the feedback loop discourages any high-frequency signals or circuit-generated oscillations due to stray capacitances between the output and the input. The diodes bias the transistors by compensating for the voltage drop at the base-emitter junction. Some

people like a higher quiescent current. You may put three diodes in series. And guess what, it does not matter much from which diode you drive the output stage, as the feedback loop will take care of any offset. The input impedance is 10K. We added the same to the noninverting input to load the opamp symmetrically. If the opamp were an ideal opamp it would not matter. But in practice it improves performance.

*I remember vividly building an amp in college only to burn the output transistors because there was high-frequency feedback. Since then, I always practice safe amplification and add a little cap parallel to the feedback resistor.*

### 9.3 CLEVER CIRCUITS

Let us revisit our multivibrator (Figure 9.7) and consider the following seemingly innocuous modification (Figure 9.14). The  $10\mu\text{F}$  capacitor in the original circuit is now replaced by a small circuit that still has the same capacitor at its core.

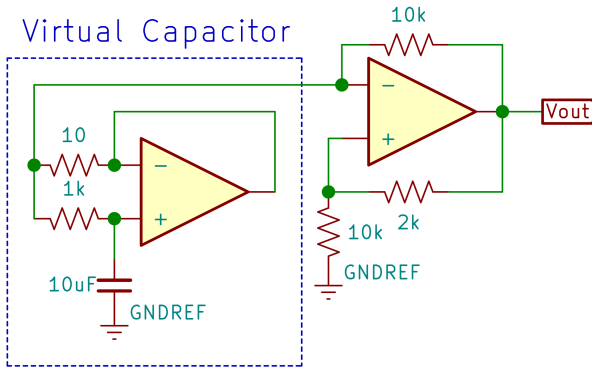


Figure 9.14: A Multivibrator with a *Virtual* Capacitor.

What would you expect the effect of this modification to be. If you build and observe the output, you once again see the output of a typical astable multivibrator (Figure 9.15).

Compare the output to that of the original circuit (Figure 9.8). Looks similar, but hold on – look what happened to the period of the output signal. It is a couple of orders of magnitude longer than the response of the original circuit. How is that possible, the capacitor is still  $10\mu\text{F}$ ?

The wickedly clever subcircuit *virtual capacitor* shown in Figure 9.14 is referred to as a *capacitance multiplier*. It makes the capacitance appear a lot more than it actually is. Many engineers simply learn this and memorize the formula (Figure 9.16).



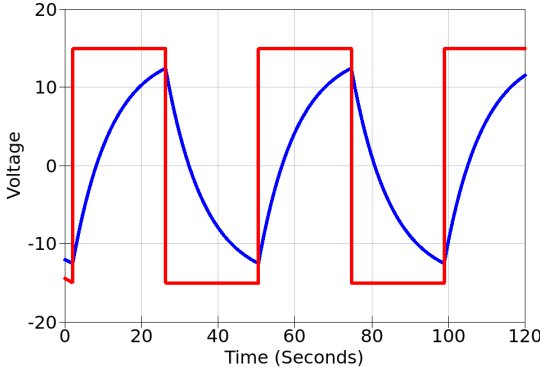


Figure 9.15: The Output of the Multivibrator with a *Virtual Capacitor*.  
(blue: voltage at the inverting input, red: output voltage  $V_{out}$ )

In our example,  $R_1 = 10 \text{ Ohm}$ . We introduce a series resistor, but it is quite small to have any significant effect on the operation of the multivibrator. The equivalent capacitance, however, is multiplied by by 100, that is,  $\frac{1k}{10}$ . Often, engineers would remember to keep  $R_1$  small,

and ignore it, thinking the capacitance is multiplied by  $\frac{R_2}{R_1}$ . That is the basic level of understanding. Let us call this *level 1* understanding. A more serious engineer might derive the equations from the basic relations. After all, the voltages and currents are all available. Such an analytical approach is made possible because we have formal systems of expressing natural phenomena. All this comes from robust *rules of inference*, which, when applied, will give us demonstrably correct results. We need not understand the physics of it, but just be careful to follow the steps rigorously. Many good engineers, textbooks, and professors would have an understanding at this level. Let us call this *level 2* understanding. Actually, with the advent of the Internet, now a *level 0* understanding is also prominent. The engineer does not know any of the details of the concept or any formula, only that there is such a thing called “capacity multiplier”. Maybe not even that. If the engineer hears the term, a look up on a search engine gives the needed information in a compact form. There may even be demo circuits that the engineer could find and consume with a simple *CTRL-V/CTRL-C* maneuver. Let us call this *level 0* understanding. None of these would be sufficient to truly appreciate the novelty of the circuit. None would allow a complete internalization of the subtleties. The analytical *level 2* might come close, but no cigar. However, engineering is about designing and producing

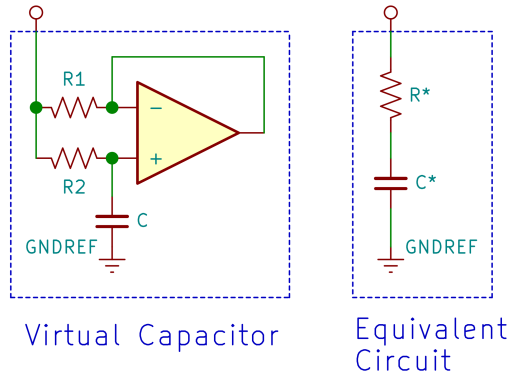


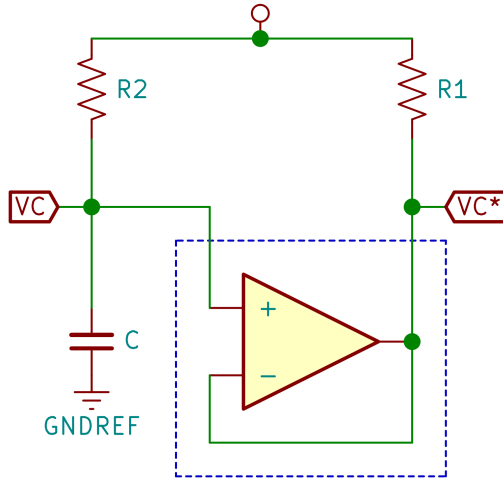
Figure 9.16: A Virtual Capacitor and its Equivalent.

$$R^* \approx R_1, \quad C^* \approx C \left( 1 + \frac{R_2}{R_1} \right)$$

artifacts. Design is a creative process that needs intuition and insight, along with imagination and a deep understanding. I can almost hear you ask, “is there a *level 3*?” Yes, indeed. There is a way to view the circuit so that it makes sense beyond memorizing or looking up the formula or deriving it from basic principles.

Any *level 3* understanding starts with a clear comprehension of the phenomenon. I often tell my students, “be one with the circuit” or “feel the circuit, Luke”. It sounds *Zen*, but it works – at least it has always worked for me as a commercial engineer who designed many products. Here, a re-draw will be worth a kilowords.

I ask that you first compare [Figure 9.17](#) to the virtual capacitor given in [Figure 9.16](#) and convince yourself that they are exactly the same. Now, let us imagine that one does not see what is in the BOX within the blue dashed lines. We then apply a voltage to the terminal and observe the response of the capacitor voltage  $V_C$  on the series subcircuit  $R_2$ - $C$ . The time constant in our multivibrator ([Figure 9.14](#)) is  $R_2 C = 1k10\mu F = 10^{-2} \text{Sec}$ . If we apply a step input, for instance, we should see the customary 63%, 86%, 95% rise during the first three time constants at node  $V_C$ . Suppose we observe the exact response at node  $V_{C^*}$ . For all practical purposes, we would assume that the two parallel branches, the  $R_2$ - $C$  and the  $R_1$ -BOX, are both resistor-capacitor voltage dividers with the same time constant. That is, it would be reasonable to think that there is a capacitor, real or virtual, in the BOX. Well,  $V_C$  and  $V_{C^*}$  are indeed at the same voltage, because the opamp senses the voltage

Figure 9.17: A *Zen* View of the Virtual Capacitor.

VC and dutifully replicates the voltage at its output,  $VC^*$ . Moreover, the capacitance  $C^*$  of this virtual capacitor must be such that it gives the exact same time constant as the first branch. That is,

$$\begin{aligned} R_2 C &= R_1 C^* \\ C^* &= C \frac{R_2}{R_1} \end{aligned} \quad (9.1)$$

Then the entire circuit, dominated by the virtual capacitor, would function roughly the same as the series resistor  $R_1$  and a virtual capacitor  $C^* = C \frac{R_2}{R_1}$ . In Figure 9.14 the capacitance of  $C$  is effectively multiplied by 100, so the  $10\mu\text{F}$  capacitor appears as a  $1\text{mF}$  capacitor. Hence the multivibrator Figure 9.14 generating long pulses with a very low frequency.

What is the benefit of *level 3* understanding? I think it takes you closer into the realm of ingenuity. Now consider the extension (Figure 9.18). If you understood the virtual capacitor, and the way voltage drivers work, you should have no trouble grasping how a small capacitor-resistor subcircuit, with the help of an opamp can imitate a large inductor.

*This is indeed funny wording. The root of the word engineer is ingenuity.*

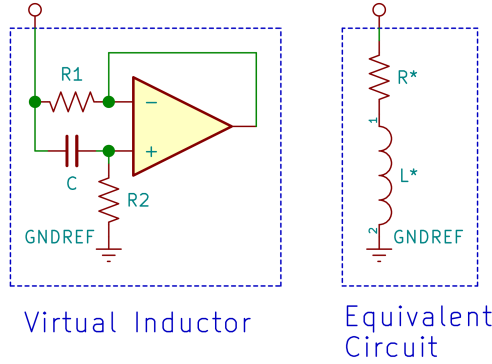


Figure 9.18: A Virtual Inductor and its Equivalent.  
 $R^* \approx R_1$ ,  $L^* \approx R_1 R_2 C$

Let us close this section with a wickedly clever circuit (Figure 9.19). It not only exemplifies engineering excellence, but also is quite versatile, and thus, may be used in many designs.

You may consider Figure 9.19 as a generalization of the virtual capacitor and virtual inductor. It shares the same architecture, as it has an input terminal and the other end is at ground voltage. If you want to make sense of this circuit, first note that the two inverting inputs are connected, so they are at the same voltage. If the circuit is operating in its linear region and no opamp is saturated, the noninverting inputs must also be at the same voltage level. That is, all opamp inputs must be at the same voltage level, as marked. The output voltages of the two opamps are also marked. We may now start computing the currents  $i_1$  to  $i_5$  through the components  $Z_1$  to  $Z_5$ . Note, however, that  $i_2 = i_3$  and  $i_4 = i_5$  (why?).

$$i_5 = \frac{V_{in}}{Z_5} = i_4 = \frac{V_1 - V_{in}}{Z_4} \tag{9.2}$$

So,

$$V_1 - V_{in} = V_{in} \frac{Z_4}{Z_5} \tag{9.3}$$

Similarly,

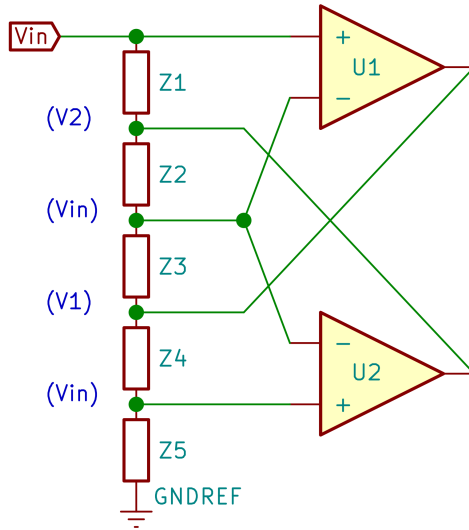


Figure 9.19: Generalized Virtual Load.

$$\begin{aligned}
 i_3 &= \frac{V_{in} - V_1}{Z_3} = i_2 = \frac{V_2 - V_{in}}{Z_2} \\
 &= V_{in} \frac{Z_4}{Z_3 Z_5} = \frac{V_2 - V_{in}}{Z_2}
 \end{aligned} \tag{9.4}$$

Which means that,

$$V_{in} - V_2 = V_{in} \frac{Z_2 Z_4}{Z_3 Z_5} \tag{9.5}$$

Finally,

$$i_1 = \frac{V_{in} - V_2}{Z_1} = V_{in} \frac{Z_2 Z_4}{Z_1 Z_3 Z_5} \tag{9.6}$$

The effective impedance  $Z_e$  of the circuit is  $\frac{V_{in}}{i_1}$ , which is,

$$Z_e = \frac{V_{in}}{i_1} = \frac{Z_1 Z_3 Z_5}{Z_2 Z_4} \tag{9.7}$$

With five different components or subcircuits to choose from, there is much latitude in this configuration. Many different loads might be created for various functionalities. You may also use this to model other systems, as in analog computation or control engineering.

Before we close this chapter, let us view just one more example from a myriad of possible ones. The circuit (Figure 9.20) is referred to as a *negative impedance converter*, often abbreviated as “NIC”.

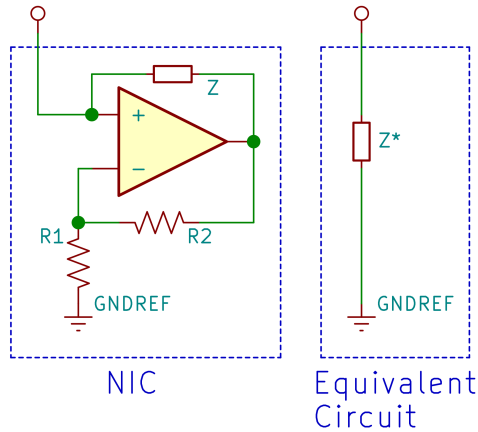


Figure 9.20: A Negative Impedance Converter and its Equivalent.

$$\text{If } R_1 = R_2, \text{ then } Z^* = -Z.$$

Let us consider the special case  $R_1 = R_2$ . The extension to  $R_1 \neq R_2$  is fairly straightforward. Rather than writing the equations, let us once again appeal to your intuition. Suppose we start at equilibrium where the input is at ground reference voltage (0V). The opamp inputs and output may remain at this equilibrium indefinitely. Now suppose that you raise the input to  $v > 0$ . Since  $v$  is applied to the noninverting input, the opamp will respond by increasing its output to  $2v$  so that the inverting input is at the same voltage  $v$  as the noninverting input. The output will be  $2v$  because  $R_1 = R_2$ , and the inverting input is always at half the voltage of the opamp output. Now look at the component with impedance  $Z$ . Its one end is at  $2v$  and its other end is at  $v$ , thus a current will flow through the component, but in reverse direction. We conclude that the circuit passes the same current through  $Z$  with a voltage difference of  $-v = 2v - v$  rather than a voltage difference of  $v = v - 0$ . Only the direction is changed. Thus, the impedance of the circuit is  $-Z$ . The name negative impedance converter fits its function perfectly.

Where may you use a NIC? If you have a component with an unwanted impedance, say the internal series resistance of an inductor, you may add the NIC to cancel out this impedance. Quite ingenious, is it not?

#### 9.4 EXPERIMENTATION AND REFLECTION

1. Considering the current sink (9.1), how would you modify the circuit so that the sink current is maintained between a lower and upper bound. Say, we want a current sink that allows any current between 1 mA and 2 mA.
2. Extend Figure 9.3 to find the maximum of three input voltages.
3. Extend Figure 9.3 to find the minimum of the two input voltages.
4. The ideal rectifier (9.2) may readily be used as a half-wave rectifier. How would you extend this concept to a full-wave rectifier?
5. Revisit Figure 9.4. Place a circuit that discharges the capacitor, so that the peak may be reset. What are the desiderata for the discharge circuit?
6. Design an astable multivibrator with a frequency of 1 kHz and a duty cycle of 20%.
7. Design and test a circuit that uses a virtual capacitor.
8. Design and test a circuit that uses a virtual inductor.
9. Design and test a tank circuit that uses a virtual capacitor and a virtual inductor to achieve a very low resonance frequency.
10. Design and test a circuit given by Figure 9.19. Compare the observed response of the circuit to the computed response.
11. Extend the circuit given by Figure 9.19 to three opamps and seven components. Obtain its effective impedance. Can you guess what the effective impedance would be in the general case with an arbitrary number of opamps?
12. Build the following circuit.

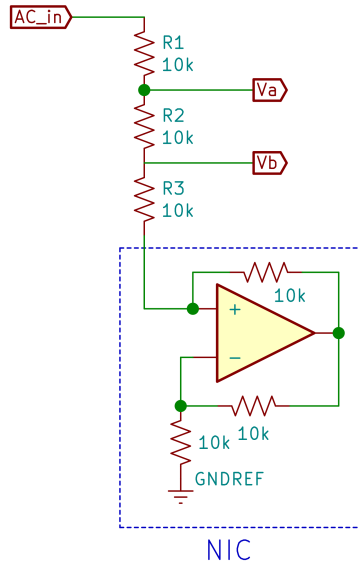


Figure 9.21: A Negative Impedance Converter Experiment.

If the NIC actually is equivalent to a  $-10\text{k}$  resistor, you should see  $V_b = 0\text{V}$  and  $V_b = \frac{AC_{in}}{2}$ . Verify that this is the case.

- Referring to [Figure 9.20](#), derive the formula for the equivalent impedance where  $R_1 \neq R_2$ .



EPILOGUE

---

This short excursion meant to provide a gut-level understanding of opamps. Along the way, we needed to visit some formalisms, some related topics such as complex numbers, transfer functions, the Laplace transform, frequency-domain ideas, and phasor algebra. By no means one could claim that we have done justice to these ideas. Our exposure was simply a glance – a stroll in the park. There is much formalism to be pursued if one must truly command these topics.

Similarly, we saw a few mundane and a few interesting opamp circuits. The set presented was perhaps the most superficial possible. There are so many more opamp applications that one may be justified to think that no single book could encapsulate its totality (although [4] and [7] come pretty close). Nonetheless, we hope we gave some insights that would contribute to engineering intuition. Rather than rely on formalisms, the approach tried to appeal to your instincts.

There is tradition that engineering subjects, as it is with the sciences, are written succinctly in what may be called a *presentation mode*. Here the topic starts with the definitions, and adhering to trusted rules of inference, you are lead to the conclusions. There is little doubt that the conclusions must hold as long as the premises were agreed upon. This leads to formalisms, which can be followed step by careful step, as does a computer. And we all know, such formalisms help us to computerize and to save us time. However, anything you do the same way a computer does seems efficient but mindless. Moreover, almost none of the concepts are discovered this way. Engineering discoveries typically happen when you perform some tasks a few times, then start observing the similarities, the patterns, the trends, the hitherto hidden structure and start developing insights. Then the results, when stated, come natural – without the need for extensive formalisms. You may not be able to prove the results, but you have inductive confidence, notwithstanding *the problem of induction*[5]. There are historical, pragmatic, and rational reasons why most textbooks prefer to follow a presentation mode exposure. In this book we attempted to follow a *discovery mode* excursion into the territory. I think this latter approach is much needed, especially in introductory engineering textbooks. We did not cover

all, and we did not prove anything here. We even told anecdotes. At times the stories may have sounded more like mythology rather than engineering. A careful blend of Feynman [3] and Campbell [2] comes to mind. Alas, the current brief exposé approaches neither in its esoterics or eloquence.

Mastering a subject seems to require repetition. This repetition is best, if each time the reader focuses on the topic at a deeper level. Otherwise, repetition would be akin to memorization. Memorization, one may argue, is the complete opposite of understanding. This book would best serve as an initial broad review of the subject, to pigeonhole the concepts into their proper places, and to gain some insights before a rigorous treatise is attempted.

Studying basic electrical engineering concepts through opamps is also a viable thought. This book was written as I was asked to teach part of a general engineering course taken by more non-electrical engineers than electrical engineers. Opamp circuits are easier to build and test, compared to transistorized circuits. An opamp amplifier could be built with just two resistors, as we have seen. With transistors, one must worry about biasing, etc. With vacuum tubes, even more.

Beyond electrical engineering, I would submit that studying opamps with a bit of flavoring from all its related topics is an excellent way to appreciate systems, natural phenomena, our instincts to use such for our own utility, and the various formalities we have developed to operationalize and internalize the concepts. As such, at the end, I suppose the topic in its totality says more about human nature than about nature itself. It demonstrates, once more, how humans have relentlessly tried and toiled to devise constructs to shape their environment, not only with physical artifacts, but also with their thoughts and formalisms, leading to intuition and instincts not to be found in nature herself.

## BIBLIOGRAPHY

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- [1] A.E.A. Araújo and D.A.V. Tonidandel. “Steinmetz and the Concept of Phasor : a Forgotten Story.” In: *Journal of Control, Automation and Electrical Systems* 24.3 (2013), pp. 388 –395.
- [2] J. Campbell, P. Cousineau, and S. L. Brown. *The Hero's Journey: The World of Joseph Campbell : Joseph Campbell on His Life and Work*. San Francisco: Harper and Row, 1990.
- [3] Richard P. Feynman. *Surely You're Joking, Mr. Feynman: Adventures of a Curious Character*. New York, NY, USA: W. W. Norton, 1985.
- [4] Paul Horowitz and Winfield Hill. *The Art of Electronics*. 3rd. USA: Cambridge University Press, 2015.
- [5] D. Hume. “Treatise of Human Nature (2nd ed).” In: ed. by L. A. Selby-Bigge. London, England: Oxford University Press, 1978.
- [6] I. Kant, C. Bennett, J. Saunders, and R. Stern. *Groundwork for the Metaphysics of Morals*. Oxford University Press, 2019.
- [7] R. Mancini. *Op Amps for Everyone: Design reference*. Amsterdam: Newnes, 2003.
- [8] P.E. Valisalo. “Hybrid Computation - Today and Tomorrow.” In: *Proc. 10th IMACS World Congress on Mathematics and Computer Simulation* (1982).
- [9] Ludwig Wittgenstein. *Philosophical Investigations*. Blackwell Publishing, 2001 (1953). ISBN: 0-631-23127-7.

## INDEX

---

3 sigma, 102

Cross-over distortion, 128

Design rule check, 116

Dimensional analysis, 24

Duty cycle, 126

Elemental circuits, 88

Energy-storing passive  
component, 86

Engineering parsimony, 91

Euler's identity, 63

Exponential decay, 102

Exponential growth, 102

Laplace transforms, 81

Level 0 understanding, 131

Level 1 understanding, 131

Level 2 understanding, 131

Level 3 understanding, 132

Negative impedance converter,  
136

Partial fractions expansion, 91

Phasor, 60

Plant, 104

Poles, 86

Pulse-width modulation, 126

Stability, 106, 116

Steady-state error, 106

Superposition, 22

Synthesis, 126

Three sigma, 102

Time constant, 24

Transfer function, 12

Voltage divider, 7

Zen, 132

Zeros, 86

## COLOPHON

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